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ORIGINAL ARTICLE

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N-dimensional versus (N–1)-dimensional connectivity testing of first-order queries to semi-algebraic sets

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Abstract This paper addresses the question whether one can determine the connectivity of a semi-algebraic set in three dimensions by looking only at two-dimensional "samples" of the set, where these samples are defined by first-order queries. The question is answered negatively for two classes of first-order queries: cartesian-product-free, and positive one-pass.

1 Introduction

Semi-algebraic sets provide a useful model for spatial datasets [7]. First-order logic over the reals (FO) then provides a basic query language for expressing queries about such spatial data. The power of FO, however, is too limited. In particular, testing whether a set in $\mathbb{R}^n$ is topologically connected is not expressible in FO for $n \geqslant 2$ (for $n = 1$ it is easily expressed).

The obvious reaction to this limitation of FO is to enrich it with an explicit operator for testing connectivity, as proposed by Giannella and Van Gucht [5] and by Benedikt et al. [2]. This operator can be applied not just to the dataset itself, but also to any set derived from the original set by an FO query.

The question now arises whether the connectivity of a set in $\mathbb{R}^n$ can be tested by testing the connectivity of a finite number of sets in $\mathbb{R}^{n-1}$, constructed from the original set by FO queries. This question is interesting because it requires to understand what is essential in a query language with connectivity tests, and what

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is redundant. Indeed, expressivity and hierarchy questions have always received a lot of attention in database theory and computational model theory [1, 4, 6].

For $n = 2$, the answer to our question is clearly negative, because connectivity in $\mathbb{R}^1$ is expressible in FO, and therefore a positive answer would imply that also connectivity in $\mathbb{R}^2$ would be expressible in FO, which we know is not true. It is intuitive to conjecture that the answer is negative for all $n \geqslant 2$.

While this conjecture in its generality remains open (and seems very hard to prove), we have proven it for two fragments of FO. In the first fragment, cartesian product is disallowed. In the second fragment, negation is disallowed, and the query must be "one pass" in a sense that can be made precise. Our treatment of the second fragment is for $n = 3$ only.

2 Preliminaries

Semi-algebraic sets A *semi-algebraic set in* $\mathbb{R}^n$ is a finite union of sets definable by conditions of the form $f_1(\mathbf{x}) = \cdots = f_k(\mathbf{x}) = 0$, $g_1(\mathbf{x}) > 0, \ldots, g_\ell(\mathbf{x}) > 0$, with $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and where $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}), g_1(\mathbf{x}), \ldots, g_\ell(\mathbf{x})$ are multivariate polynomials in the variables $x_1, \ldots, x_n$ with real coefficients.

Semi-algebraic sets form a very robust class; for example, any set definable by a formula with quantifiers in first-order logic over the reals is semi-algebraic (i.e., definable also without quantifiers; this is the Tarski–Seidenberg principle [3]).

Relational algebra To express first-order queries about a set S in $\mathbb{R}^n$, we use not the formalism of first-order logic, but the equivalent formalism of relational algebra expressions (RAEs). These are inductively defined as follows. The symbol S is a RAE, of *arity* n. Any constant semi-algebraic set in $\mathbb{R}^k$, for any k, is a RAE of arity k. If e_1 and e_2 are REAs of arities k_1 and k_2 respectively, then the cartesian product $(e_1 \times e_2)$ is a RAE of arity $k_1 + k_2$, and provided that $k_1 = k_2 = k$, the union $(e_1 \cup e_2)$, the intersection $(e_1 \cap e_2)$ and the difference $(e_1 - e_2)$ are RAEs of arity k. Finally, if e is a RAE of arity k, and $i_1, \ldots, i_p \in \{1, \ldots, k\}$, then the projection $\pi_{i_1,\ldots,i_p}(e)$ is a RAE of arity p.

When applied to a given set A in $\mathbb{R}^n$, a RAE e of arity k evaluates in the natural way to a set $e(A)$ in $\mathbb{R}^k$. When A is semi-algebraic, $e(A)$ is too, by the Tarski–Seidenberg principle.

Notation We will use the following notations.

- The topological closure of a set $A \subseteq \mathbb{R}^n$ is denoted by $\mathrm{cl}(A)$, its interior is denoted by $\mathrm{int}(A)$ and its boundary $\mathrm{cl}(A) - \mathrm{int}(A)$ is denoted by $\mathrm{bd}(A)$.
- The n-dimensional closed unit ball centered around the origin is denoted by $\blacksquare$; the n-dimensional unit sphere centered around the origin by $\square$; and the union $\square \cup \{\mathbf{0}\}$ by $\boxdot$.
- The set of affine transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ (compositions of a scaling and a translation) is denoted by $\mathbb{A}$.

3 Cartesian-product-free queries

A RAE is called *cartesian-product-free* if it does not use cartesian product. An example of such a RAE is

$$\pi_{1,2}((S \cap \Gamma_1) \cup (\Gamma_2 - S)) - \pi_{1,3}(S \cup \Gamma_3)$$

where Γ_1, Γ_2 and Γ_3 can be arbitrary semi-algebraic sets in $\mathbb{R}^3$ and S is ternary (i.e., stands for a set in $\mathbb{R}^3$).

In this section, we prove that the connectivity of a semi-algebraic set in $\mathbb{R}^3$ cannot be determined by sampling it using a finite number of binary cartesian-product-free RAEs.

Theorem 1 *Let S range over sets in $\mathbb{R}^3$. For any finite collection $e_1, \dots, e_\ell$ of binary cartesian-product-free RAEs over S, there exist two semi-algebraic sets A and B in $\mathbb{R}^3$ such that*

1. *A is connected;*
2. *B is disconnected;*
3. *$e_i(A) = e_i(B)$ for $i = 1, \dots, \ell$.*

Toward the proof, we start with the following observation.

Lemma 1 *Let $\Lambda_0, \Lambda_1, \dots, \Lambda_k$ be nonempty semi-algebraic sets in $\mathbb{R}^3$, where Λ_0 is open. Then there exists a partition $\{I, J\}$ of $\{1, \dots, k\}$ and an open semi-algebraic set $V \subseteq \Lambda_0$ such that*

- *$V \subseteq \Lambda_i$ for $i \in I$, and*
- *$V \cap \Lambda_j = \varnothing$ for $j \in J$.*

Proof By induction on k. If $k = 0$, set $I = \{0\}$, $J = \varnothing$, and $V = \Lambda_0$.

If $k > 0$, consider the set $\{\Lambda_0, \Lambda_1, \dots, \Lambda_{k-1}\}$. Then by the induction hypothesis, there is a partition $\{I', J'\}$ of $\{1, \dots, k-1\}$ and an open set $V' \subseteq \Lambda_0$ satisfying the condition as stated in the lemma for $k - 1$. Since $V' = (V' \setminus \Lambda_k) \cup (V' \cap \Lambda_k)$, since $\dim V' = 3$, and since $\dim(A \cup B) = \max\{\dim A, \dim B\}$ for semi-algebraic sets A and B, at least one of the following two cases occurs:

1. $\dim(V' \setminus \Lambda_k) = 3$, in which case we choose V an open subset of $V' \setminus \Lambda_k$, and set $I = I'$ and $J = J' \cup \{k\}$.
2. $\dim(V' \cap \Lambda_k) = 3$, in which case we choose V an open subset of $V' \cap \Lambda_k$, and set $I = I' \cup \{k\}$ and $J = J'$. □

The following lemma is the crucial element in our proof of the theorem.

Lemma 2 *For a given open semi-algebraic set $U \subseteq \mathbb{R}^3$, and any ternary cartesian-product-free RAE e, there exists an open set $V \subseteq U$ such that e is equivalent to an expression of one of the four possible forms*

$$\Gamma, \quad S, \quad S \cup \Gamma, \quad \Gamma - S$$

on all sets $S \subseteq V$, where Γ denotes a constant set in $\mathbb{R}^3$. Moreover, in the last form, V is included in the interior of Γ.

Proof Since both the input S to e and the output of e are ternary, and e is cartesian-product-free, e must be projection-free as well. By rewriting $(e_1 \cap e_2)$ as $(e_2 - (e_2 - e_1))$ we can ignore the intersection operator. We now proceed by induction on the structure of e. The base cases where e is S or e is constant are already in the right form.

For the cases $e = (e_1 \cup e_2)$ and $e = (e_1 - e_2)$, by induction we can find an open set $V_1 \subseteq U$ such that e_1 has one of the four possible forms within V_1, and we can

Table 1 Proof of Lemma 2, possibilities for $e_1 \cup e_2$

$\cup$	S	Γ_2
S	S	$S \cup \Gamma_2$
Γ_1	$S \cup \Gamma_1$	$\Gamma_1 \cup \Gamma_2$
$S \cup \Gamma_1$	$S \cup \Gamma_1$	$S \cup (\Gamma_1 \cup \Gamma_2)$
$\Gamma_1 - S$	Γ_1	$\{\Gamma_1 \cup \Gamma_2, (\Gamma_1 \cup \Gamma_2) - S\}$
$\cup$	$S \cup \Gamma_2$	$\Gamma_2 - S$
S	$S \cup \Gamma_2$	Γ_2
Γ_1	$S \cup (\Gamma_1 \cup \Gamma_2)$	$\{\Gamma_1 \cup \Gamma_2, (\Gamma_1 \cup \Gamma_2) - S\}$
$S \cup \Gamma_1$	$S \cup (\Gamma_1 \cup \Gamma_2)$	$S \cup (\Gamma_1 \cup \Gamma_2)$
$\Gamma_1 - S$	$S \cup (\Gamma_1 \cup \Gamma_2)$	$(\Gamma_1 \cup \Gamma_2) - S$

Table 2 Proof of Lemma 2, possibilities for $e_1 - e_2$

$-$	S	Γ_2
S	$\varnothing$	$\{\varnothing, S\}$
Γ_1	$\Gamma_1 - S$	$\Gamma_1 - \Gamma_2$
$S \cup \Gamma_1$	$\{\Gamma_1, \Gamma_1 - S\}$	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) \cup S\}$
$\Gamma_1 - S$	$\Gamma_1 - S$	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) - S\}$
$-$	$S \cup \Gamma_2$	$\Gamma_2 - S$
S	$\varnothing$	S
Γ_1	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) - S\}$	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) \cup S\}$
$S \cup \Gamma_1$	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) - S\}$	$S \cup (\Gamma_1 - \Gamma_2)$
$\Gamma_1 - S$	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) - S\}$	$\{\Gamma_1 - \Gamma_2, (\Gamma_1 - \Gamma_2) - S\}$

further find an open set $V_2 \subseteq V_1$ such that e_2 has one of the four possible forms within V_2. This means that we have to consider $2 \times 4 \times 4$ possibilities (actually less, as there are symmetries), shown in Tables 1 and 2.

Take, for example, $e = (\Gamma_1 - S) \cup \Gamma_2$. By applying Lemma 1 to $\Lambda_0 = V_2$ and $\Lambda_1 = \Gamma_1$, we get a $V \subseteq V_2$ such that either $V \subseteq \Gamma_1$ or $V \cap \Gamma_1 = \varnothing$. In the latter case, e is equivalent to $\Gamma_1 \cup \Gamma_2$ within V. In the former case, e is equivalent to $(\Gamma_1 \cup \Gamma_2) - S$ within V, and we can always shrink V a bit so that it is included in the interior of $\Gamma_1 \cup \Gamma_2$, in accordance with the statement of the lemma. In both cases e is in a desired form. We summarize this in the corresponding entry in Table 1. All other entries in the tables are proven similarly, or are trivial. □

We are now ready for the proof of Theorem 1:

Proof A binary cartesian-product-free RAE e over ternary S can be viewed as an expression built up, using the operators $\cup$ and $-$, from binary constant sets and binary projections of ternary cartesian-product-free RAEs. If $\pi_{i,j}(c)$ is such a projection occurring in e, we call c a *component* of e.

By a series of applications of Lemma 2, we can get all components of all the given binary expressions $e_1, \ldots, e_\ell$ in one of the four normal forms mentioned in the lemma. The first application starts with $U = \mathbb{R}^3$, and every next application takes as U the V produced by the previous application. Within the V produced by the final application, all components are in normal form.

Choose $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset V$, and consider the sets $A = \tau(\square)$ (which is connected) and $B = \tau(\boxdot)$ (which is disconnected). Now any binary projection $\pi_{i,j}$ of a component c in normal form yields the same result whether applied to A or to B. Indeed, if c is of the form Γ, S, or $S \cup \Gamma$ this is clear; if c is of the form $\Gamma - S$ then we recall that Lemma 2 guarantees that V is fully included in the interior of Γ, so $\pi_{i,j}(\Gamma - S) = \pi_{i,j}(\Gamma)$.

We can thus conclude that $e_i(A) = e_i(B)$ for $i = 1, \ldots, \ell$ as desired. □

For simplicity of exposition, in this section, we have stated and proved Theorem 1 in three dimensions only. However, the argument readily generalizes to prove for any $n > 2$ that the connectivity of a semi-algebraic set in $\mathbb{R}^n$ cannot be determined by sampling it using a finite number of $n-1$-ary cartesian-product-free RAEs.

Theorem *Let $n > 2$, and let S range over sets in $\mathbb{R}^n$. For any finite collection $e_1, \ldots, e_\ell$ of $n-1$-ary cartesian-product-free RAEs over S, there exist two semi-algebraic sets A and B in $\mathbb{R}^n$ such that*

1. *A is connected;*
2. *B is disconnected;*
3. *$e_i(A) = e_i(B)$ for $i = 1, \ldots, \ell$.*

4 Positive one-pass queries

A RAE is called *positive one-pass* if it does not use the difference operator, and mentions S only once. An example is

$$\pi_{3,5}(\Lambda_1 \cup (\Lambda_2 \cap (S \times \mathbb{R}^2)))$$

where S is ternary, and Λ_1 and Λ_2 are arbitrary semi-algebraic sets in $\mathbb{R}^5$. As a matter of fact, this example is very representative, in view of the following:

Lemma 3 *Every binary positive one-pass RAE can be written in the form*

$$\pi_{i_1,i_2}(\Lambda_1 \cup (\Lambda_2 \cap (S \times \mathbb{R}^k))).$$

More generally, it can be verified by induction that every p-ary positive one-pass RAE can be written in the form of the above lemma, with π_{i_1,i_2} replaced by $\pi_{i_1,\ldots,i_p}$.

In this section, we prove that the connectivity of a semi-algebraic set in $\mathbb{R}^3$ cannot be determined by sampling it using a finite number of binary positive one-pass RAEs.

Theorem 2 *Let S range over sets in $\mathbb{R}^3$. For any finite collection $e_1, \ldots, e_\ell$ of binary positive one-pass RAEs over S, there exist two semi-algebraic sets A and B in $\mathbb{R}^3$ such that*

1. *A is connected;*
2. *B is disconnected;*
3. *$e_i(A) = e_i(B)$ for $i = 1, \ldots, \ell$.*

The following lemma essentially proves the theorem.

Lemma 4 *For a given open semi-algebraic set* $U \subseteq \mathbb{R}^3$*, any semi-algebraic sets* Λ_1 *and* Λ_2 *in* $\mathbb{R}^{3+k}$*, and any* $i_1, i_2 \in \{1, 2, 3, \ldots, k+3\}$*, we can always find an open set* $V \subseteq U$ *such that for any* $\tau \in \mathbb{A}$ *with* $\tau(\blacksquare) \subset V$*,*

$$\pi_{i_1,i_2}(\Lambda_1 \cup (\Lambda_2 \cap (\tau(\square) \times \mathbb{R}^k))) = \pi_{i_1,i_2}(\Lambda_1 \cup (\Lambda_2 \cap (\tau(\blacksquare) \times \mathbb{R}^k))).$$

Assuming this lemma, we can give the proof of Theorem 2:

Proof By a series of applications of Lemma 4, we obtain a V such that for any $\tau \in \mathbb{A}$ for which $\tau(\blacksquare) \subset V$, we have $e_i(\tau(\square)) = e_i(\tau(\blacksquare))$ for $i = 1, \ldots, \ell$. Since every e_i is positive (does not use the difference operator), every e_i is monotone with respect to the subset order. Hence, $e_i(\tau(\square)) \subseteq e_i(\tau(\boxdot)) \subseteq e_i(\tau(\blacksquare))$ and thus $e_i(\tau(\square)) = e_i(\tau(\boxdot))$. Taking $A = \tau(\square)$ and $B = \tau(\boxdot)$ thus proves the theorem. □

To prove Lemma 4 we will use the regular cell decomposition of semi-algebraic sets, whose definition we recall next. A function $f : C \to \mathbb{R}$, where $C \subseteq \mathbb{R}^n$, is called *regular* if it is continuous and for each $i \in \{1, \ldots, n\}$ either strictly increasing, strictly decreasing, or constant in the i^{th} coordinate. (Which of these three cases holds may depend on i.) Also, we call f semi-algebraic if its graph is semi-algebraic.

We define a *regular cell* by induction on the number of dimensions. Regular cells in $\mathbb{R}$ are singletons $\{a\}$, or open intervals (a, b), $(-\infty, a)$, or $(a, +\infty)$. Now assume that $C \subseteq \mathbb{R}^n$ is a regular cell, and $f, g : C \to \mathbb{R}$ are regular semi-algebraic functions on C, with $f(\mathbf{x}) < g(\mathbf{x})$ for all $\mathbf{x} \in C$. Then the sets $\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in C\}$ and $\{(\mathbf{x}, r) \mid \mathbf{x} \in C, f(\mathbf{x}) < r < g(\mathbf{x})\}$ are regular cells in $\mathbb{R}^{n+1}$. In the latter case, f can be $-\infty$, and g can be ∞.

A *regular cell decomposition of* $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into a finite number of regular cells. This is also defined by induction on n. A regular decomposition of $\mathbb{R}$ is just any finite partition of $\mathbb{R}$ in regular cells. For $n > 1$, a regular cell decomposition of $\mathbb{R}^n$ is a finite partition $\{\mathcal{S}_1, \ldots, \mathcal{S}_k\}$ of $\mathbb{R}^n$ in regular cells such that $\{\pi(\mathcal{S}_1), \ldots, \pi(\mathcal{S}_k)\}$ is a regular cell decomposition of $\mathbb{R}^{n-1}$. Here, $\pi : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ is the natural projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$.

Let A be a semi-algebraic set in $\mathbb{R}^n$. A regular cell decomposition of $\mathbb{R}^n$ is said to be *compatible* with A if A is a union of regular cells from this decomposition.

Fact ([8]) *For every semi-algebraic set* A *in* $\mathbb{R}^n$ *there exists a regular cell decomposition of* $\mathbb{R}^n$ *compatible with* A*.*

Toward the proof of Lemma 4, we start with the following sequence of observations.

Lemma 5 *Let* $A \subseteq \mathbb{R}^3$ *be a compact semi-algebraic set and let* $f : A \to \mathbb{R}$ *be a regular function. Then* $\min_A f = \min_{\text{bd}(A)} f$ *and* $\max_A f = \max_{\text{bd}(A)} f$*. If moreover the boundary* $\text{bd}(A)$ *is connected, then* $f(\text{bd}(A))$ *equals the interval* $[\min_A f, \max_A f]$.

Proof Since A is closed, $\text{bd}(A) \subseteq A$ and thus $\min_A f \geqslant \min_{\text{bd}(A)} f$. To show the reverse inequality, we need to find for any point in $A - \text{bd}(A)$ another point in $\text{bd}(A)$ with the same or lower f-value. Take a point $(x, y, z) \in A - \text{bd}(A)$, and shoot a straight ray out of that point in any direction. Since A is bounded, the ray will intersect $\mathrm{bd}(A)$. Let us focus on the two rays orthogonal to the xy plane. If f is strictly decreasing in z, shoot the ray in increasing z direction to obtain an intersection point with $\mathrm{bd}(A)$ with lower f-value as desired. If f is strictly increasing, follow the converse direction, and if f is constant, any direction will do to find a point in $\mathrm{bd}(A)$ with the same f-value. The equality $\max_A f = \max_{\mathrm{bd}(A)} f$ is proven in the same way.

Now assume $\mathrm{bd}(A)$ is connected. Choose $\mathbf{x}_{\max} \in \mathrm{bd}(A)$ with maximal f-value, and choose $\mathbf{x}_{\min} \in \mathrm{bd}(A)$ with minimal f-value. Since for semi-algebraic sets, connectivity coincides with path connectivity [3], there is a continuous path $\gamma : [0, 1] \to \mathrm{bd}(A)$ such that $\gamma(0) = \mathbf{x}_{\min}$ and $\gamma(1) = \mathbf{x}_{\max}$. Since f is continuous, so is the composition $f \circ \gamma$. Since $[0, 1]$ is closed and connected, $f \circ \gamma([0, 1])$ must be a closed and connected set in $\mathbb{R}$, and must therefore equal the interval $[\min_A f, \max_A f]$. □

As a consequence, we have that

Lemma 6 *Let W be an open set in $\mathbb{R}^3$ and let f be a regular function of W. Then for any compact and connected semi-algebraic set $A \subseteq W$ and for $i_1 \in \{1, 2, 3\}$, we have that*

$$\pi_{i_1,4}(\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in A\}) = \pi_{i_1,4}(\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in \mathrm{bd}(A)\}).$$

Proof Assume that $i_1 = 3$, the other cases are analogous. We only need to prove the inclusion

$$\pi_{3,4}(\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in A\}) \subseteq \pi_{3,4}(\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in \mathrm{bd}(A)\}),$$

the other direction being trivial. Take an arbitrary element $(z_0, f(x_0, y_0, z_0)) \in \pi_{3,4}(\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in A\})$. Since $\{(x, y, z) \in A \mid z = z_0\}$ is compact with connected boundary, we can apply Lemma 5 to obtain $(x_1, y_1, z_0) \in \mathrm{bd}(A)$ with $f(x_1, y_1, z_0) = f(x_0, y_0, z_0)$. Hence, $(z_0, f(x_0, y_0, z_0)) \in \pi_{3,4}(\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in \mathrm{bd}(A)\})$ as desired. □

We also observe the following useful property of regular functions.

Lemma 7 *Let W be an open set in $\mathbb{R}^3$ and let f, g be two regular functions on W. Then there exists a function $h(\mathbf{x}, t)$ such that (i) for any $t \in (0, 1)$, $h(\cdot, t)$ is regular on W; and (ii) for any semi-algebraic set $A \subseteq W$,*

$$\{(\mathbf{x}, u) \mid \mathbf{x} \in A \wedge f(\mathbf{x}) < u < g(\mathbf{x})\} = \bigcup_{t \in (0,1)} \{(\mathbf{x}, h(\mathbf{x}, t)) \mid \mathbf{x} \in A\}. \tag{1}$$

Proof We say that f and g behave in an *opposite way* if when f is increasing (decreasing) in the ith coordinate, then g is decreasing (increasing) in the ith coordinate.

Case 1: Suppose that f and g are not behaving in an opposite way in any of the coordinates. Then it is easily verified that $h(x, y, z, t) = tf(x, y, z) + (1 - t)g(x, y, z)$ is a regular function for any $t \in (0, 1)$. Moreover, the equality (1) trivially holds.

Case 2: Suppose now that f and g behave in an opposite way for some coordinates; let $I \subseteq \{1, 2, 3\}$ denote the coordinates for which f and g do so. Let $i \in I$ and $\mathbf{x} = (x, y, z)$. We denote by $\mathbf{x} + \mathbf{i}$ the translation of $\mathbf{x}$ by a vector $\mathbf{i}$ along the ith coordinate. We define

$$M_i(f)(\mathbf{x}) = \sup_{\mathbf{x}+\mathbf{i} \in W} f(\mathbf{x} + \mathbf{i}),$$

and similarly,

$$m_i(g)(\mathbf{x}) = \inf_{\mathbf{x}+\mathbf{i} \in W} g(\mathbf{x} + \mathbf{i}).$$

For each $\mathbf{x} \in W$, we have that $f(\mathbf{x}) \leqslant M_i(f)(\mathbf{x})$, $m_i(g)(\mathbf{x}) \leqslant g(\mathbf{x})$ and $M_i(f)(\mathbf{x}) \leqslant m_i(g)(\mathbf{x})$. Moreover, for each $\mathbf{x} \in W$, $h_i(\mathbf{x})$, $M_i(f)$ and $m_i(g)$ behave in an opposite way in the coordinates $I \setminus \{i\}$, are constant in the ith coordinate and $M_i(f)$ (resp. $m_i(g)$) behaves as f (resp. g) in the other coordinates.

Let $I = I \setminus \{i\}$. We now repeat the above for $M_i(f)$, $m_i(g)$ and I until $I = \varnothing$. At the end of this process, we have two regular functions $M(f)$ and $m(g)$ which do not behave in an opposite way in any coordinate, which are constant in the coordinates in the original set I, and such that $M(f)$ and $m(g)$ behave just like f (or, similarly g) in the other coordinates. Moreover, for each $\mathbf{x} \in W$ we have that $f(\mathbf{x}) \leqslant M(f)(\mathbf{x}) \leqslant m(g)(\mathbf{x}) \leqslant g(\mathbf{x})$.

We now define

$$p(\mathbf{x}) = M(f)(\mathbf{x}) + \frac{1}{2}(m(g)(\mathbf{x}) - M(f)(\mathbf{x})),$$

and apply Case 1 to the pairs f and p, and p and g. Hence, for each pair we obtain a function $h_i(x, y, z, t)$, $t \in (0, 1)$, $i = 1, 2$. Finally, the desired function is defined as $h(x, y, z, t) = h_1(x, y, z, 2t)$ for $0 < t \leqslant 1/2$ and $h(x, y, z, t) = h_2(x, y, z, 2t - 1/2)$ for $1/2 < t < 1$. □

We are now ready to embark on the proof of Lemma 4:

Proof First note that $\pi_{i_1,i_2}(\Lambda_1 \cup (\Lambda_2 \cap (S \times \mathbb{R}^k)))$ is equivalent to $\pi_{i_1,i_2}(\Lambda_1) \cup \pi_{i_1,i_2}(\Lambda_2 \cap (S \times \mathbb{R}^k))$. So we may focus on expressions of the form

$$\pi_{i_1,i_2}(\Lambda \cap (S \times \mathbb{R}^k)). \tag{2}$$

We only need to prove the inclusion

$$\pi_{i_1,i_2}(\Lambda \cap (\tau(\blacksquare) \times \mathbb{R}^k)) \subseteq \pi_{i_1,i_2}(\Lambda \cap (\tau(\square) \times \mathbb{R}^k)), \tag{3}$$

the other direction being trivial. The proof consists of several cases depending on the indices i_1, i_2.

Case 1: $i_1, i_2 \in \{1, 2, 3\}$

Expression (2) is equivalent to $\pi_{i_1,i_2}(E)$, where E is $\pi_{1,2,3}(\Lambda \cap (S \times \mathbb{R}^k))$. Applying Lemma 1 to $\Lambda_0 = \mathbb{R}^3$ and $\Lambda_1 = \pi_{1,2,3}(\Lambda)$, we get an open set V such that we are in one of the following two cases.

1. $V \cap \Lambda_1 = \varnothing$.
   Within V, expression E, and hence also (2), reduces to the empty set, so the inclusion (3) to be proven trivially holds within V.
2. $V \subset \Lambda_1$.
   Within V, expression E now reduces to S, so expression (2) reduces to $\pi_{i_1,i_2}(S)$. In particular this holds for both $S = \tau(\blacksquare)$ and $S = \tau(\square)$, where $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset V$. Since $\pi_{i_1,i_2}(\tau(\blacksquare)) = \pi_{i_1,i_2}(\tau(\square))$, the inclusion (3) holds within V.

Case 2: $i_1 \in \{1, 2, 3\}, i_2 \notin \{1, 2, 3\}$

Expression (2) is now equivalent to $\pi_{i_1,4}(E)$, where E now is $\pi_{1,2,3,i_2}(\Lambda \cap (S \times \mathbb{R}^k))$. Applying Lemma 1 to $\Lambda_0 = \mathbb{R}^3$ and $\Lambda_1 = \pi_{1,2,3}(\Lambda)$, we get an open set $V^{(0)}$ such that we are in one of the following cases.

1. $V^{(0)} \cap \Lambda_1 = \varnothing$.
   Within $V^{(0)}$, expression E, and hence also (2), reduces to the empty set, so the inclusion (3) holds within $V^{(0)}$.
2. $V^{(0)} \subset \Lambda_1$.
   Within $V^{(0)}$, expression E now reduces to $A \cap (S \times \mathbb{R})$, with $A = \pi_{1,2,3,i_2}(\Lambda)$. Consider a regular cell decomposition of $\mathbb{R}^4$ compatible with A, and write the projection of this decomposition onto $\mathbb{R}^3$ as $\{C_1, \ldots, C_\ell\}$. Applying Lemma 1 to $\Lambda_0^{(1)} = V^{(0)}$, and $\Lambda_i^{(1)} = C_i \cap V^{(0)}$ for $i = 1, \ldots, \ell$, we get an open set $V^{(1)} \subset V^{(0)}$ contained in a unique cell C_j. Due to our regular cell decomposition, in particular the parts based on C_j, within $V^{(1)}$ the expression $E = A \cap (S \times \mathbb{R})$ can now be written as a union of sets of the form

   $$E_1 = \{(x, y, z, v) \mid (x, y, z) \in S \wedge v = f(x, y, z)\}$$

   or

   $$E_2 = \{(x, y, z, v) \mid (x, y, z) \in S \wedge f(x, y, z) < v < g(x, y, z)\},$$

   where f and g are regular functions.
   Assume that $i_1 = 3$ (the cases $i_1 = 1, 2$ are analogous) so that the inclusion (3) to be proven becomes $\pi_{3,4}(E(\tau(\blacksquare))) \subseteq \pi_{3,4}(E(\tau(\square)))$. Since the projection of a union is the union of the projections, we can restrict attention to the cases $E = E_1$ and $E = E_2$.
   (a) $E = E_1$. Consider Lemma 6 with $W = V^{(1)}$ and $f = f(x, y, z)$ the regular function in the definition of E_1. The result then follows by taking $A = \tau(\blacksquare)$ where $\tau \in \mathbb{A}$ is such that $\tau(\blacksquare) \subset W$.

(b) $E = E_2$. Consider Lemma 7 with $W = V^{(1)}$, $f = f(x, y, z)$ and $g = g(x, y, z)$ the regular functions in the definition of E_2. It follows that E_2 can be written as the union of (uncountably many) sets of the form E_1. We can treat each of these E_1-like sets by applying Lemma 6 with $W = V^{(1)}$ and $f = h(x, y, z, t)$, $t \in (0, 1)$. Here, $h(x, y, z, t)$ is the function stated in Lemma 7. Since W is independent of t, it is sufficient to choose a single $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset W$.

Case 3: $i_1, i_2 \notin \{1, 2, 3\}$

Expression (2) is now equivalent to $\pi_{4,5}(E)$, where E is $\pi_{1,2,3,i_1,i_2}(\Lambda \cap (S \times \mathbb{R}^k))$. Applying, as always, Lemma 1 to $\Lambda_0 = \mathbb{R}^3$ and $\Lambda_1 = \pi_{1,2,3}(\Lambda)$, we get an open set $V^{(0)}$ such that either $V^{(0)} \cap \Lambda_1 = \varnothing$ or $V^{(0)} \subset \Lambda_1$.

If $V^{(0)} \cap \Lambda_1 = \varnothing$, within $V^{(0)}$, expression E, and hence also (2), reduces to the empty set, so the inclusion (3) holds within $V^{(0)}$.

So we can assume that $V^{(0)} \subset \Lambda_1$. Within $V^{(0)}$, expression E now reduces to $A \cap (S \times \mathbb{R}^2)$, with $A = \pi_{1,2,3,i_1,i_2}(\Lambda)$. Consider a regular cell decomposition of $\mathbb{R}^5$ compatible with A, and write the projection of this decomposition onto $\mathbb{R}^3$ as $\{C_1, \ldots, C_\ell\}$. Applying Lemma 1 to $\Lambda_0^{(1)} = V^{(0)}$, and $\Lambda_i^{(1)} = C_i \cap V^{(0)}$ for $i = 1, \ldots, \ell$, we get an open set $V^{(1)} \subset V^{(0)}$ contained in a unique cell C_j. Due to our regular cell decomposition, in particular the parts based on C_j, within $V^{(1)}$ the expression $E = A \cap (S \times \mathbb{R}^2)$ can now be written as a union of sets of the form

$$
\begin{aligned}
E_1 &= \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = f(x, y, z) \wedge v = g(x, y, z, u)\},\\
E_2 &= \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = f(x, y, z)\\
&\qquad\qquad \wedge g_1(x, y, z, u) < v < g_2(x, y, z, u)\},\\
E_3 &= \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge f_1(x, y, z) < u < f_2(x, y, z)\\
&\qquad\qquad \wedge v = g(x, y, z, u)\}, \text{ or}\\
E_4 &= \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge f_1(x, y, z) < u < f_2(x, y, z)\\
&\qquad\qquad \wedge g_1(x, y, z, u) < v < g_2(x, y, z, u)\},
\end{aligned}
$$

where f, f_1, f_2, g, g_1, and g_2 are regular functions.

We need to prove $\pi_{4,5}(E(\tau(\blacksquare))) \subseteq \pi_{4,5}(E(\tau(\square)))$. Since the projection of an union is the union of the projections, we can restrict attention to the cases $E = E_1$, $E = E_2$, $E = E_3$, and $E = E_4$.

1. $E = E_1$.
   (a) If f is constant, with value u_0, we observe that for $i = 3$ (the cases $i = 1, 2$ are analogous) $\pi_{i,4,5}(E_1) = \{(z, u_0, v) \mid (z, v) \in \pi_{i_1,5}(E_1)\}$. From Lemma 6 with $W = V^{(1)}$ and $f = g(x, y, z, u_0)$ the regular function in the definition of E_1, we know that for any $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset W$, $\pi_{i,5}(E_1(\tau(\blacksquare))) = \pi_{i,5}(E_1(\tau(\square)))$. Hence, we have that $\pi_{i,4,5}(E_1(\tau(\blacksquare))) = \pi_{i,4,5}(E_1(\tau(\square)))$ already, so this certainly holds when projecting even further on the fourth and fifth coordinate.
   (b) Now assume that f is not constant in x; the cases y and z are analogous. Look at the projection $\pi_{2,3,4,5}(E_1)$:

$$\{(y, z, u, v) \mid \exists x((x, y, z) \in S \wedge u = f(x, y, z) \wedge v = g(x, y, z, u))\}.$$

This set can be written as

$$E_1' = \{(y, z, u, v) \mid (y, z, u) \in h(S) \wedge v = k(y, z, u)\},$$

where $h : (x, y, z) \mapsto (y, z, f(x, y, z))$, and

$$k : (y, z, u) \mapsto g\big(h_x^{-1}(y, z, u), y, z, u\big),$$

where h_x^{-1} is the function defined by $h(h_x^{-1}(y, z, u), y, z) = (y, z, u)$. This inverse function exists; in fact, because f is regular and non-constant in x, h is a homeomorphism within $V^{(1)}$.

Within $W^{(0)} = h(V^{(1)})$, we can find an open set $W^{(1)} \subset W^{(0)}$ such that k is regular. Since h is a homeomorphism, we also can find an open set $V^{(2)} \subset V^{(1)}$ such that $h(V^{(2)}) \subset W^{(1)}$.

Since $\pi_{4,5}(E_1)$ reduces to $\pi_{3,4}(E_1')$, we can apply Lemma 6 with $W = h(V^{(2)})$ and $f = k(y, z, u)$. The result then follows by taking $\tau \in \mathbb{A}$ such that $h(\tau(\blacksquare)) \subset W$.

2. $E = E_2$.

(a) If f is constant, with value u_0, then we observe that

$$\begin{aligned} E_2 = \{(x, y, z, v) \mid (x, y, z) \in S \\ \wedge g_1(x, y, z, u_0) < v < g_2(x, y, z, u_0)\} \times \{u \mid u = u_0\}. \end{aligned}$$

Note that for a fixed u_0, $g_1(x, y, z, u_0)$ and $g_2(x, y, z, u_0)$ are regular functions. Hence, by applying Lemma 7 with $W = V^{(1)}$ and $f = g_1(x, y, z, u_0)$ and $g = g_2(x, y, z, u_0)$, we know that we can write E_2 as the union of sets of the form $E_1 \times \{u \mid u = u_0\}$. We can treat each of these E_1-like sets by applying Lemma 6 with $W = V^{(1)}$ and $f = h(x, y, z, t)$, $t \in (0, 1)$. Here, $h(x, y, z, t)$ is the function stated in Lemma 7. Since W is independent of t, it is sufficient to choose a single $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset W$.

(b) Now assume f is not constant in x; the cases y and z are analogous. Look at the projection $\pi_{2,3,4,5}(E_2)$:

$$\begin{aligned} \{(y, z, u, v) \mid \exists x((x, y, z) \in S \wedge u = f(x, y, z) \\ \wedge g_1(x, y, z, u) < v < g_2(x, y, z, u)\}. \end{aligned}$$

As in Case 3.1(b), we can find $V^{(2)} \subset V^{(1)}$ such that $\pi_{2,3,4,5}(E_2)$ can be written as

$$E_2' = \{(y, z, u, v) \mid (y, z, u) \in h(S) \wedge k_1(y, z, u) < v < k_2(y, z, u)\},$$

where $h : (x, y, z) \mapsto (y, z, f(x, y, z))$, and $k_i(y, z, u)$, $i = 1, 2$ are regular functions on $h(V^{(2)})$. By applying Lemma 7 with $W = h(V^{(2)})$ and $f = k_1(y, z, u)$ and $g = k_2(y, z, u)$, we know that E_2' can be written as the union of sets of the form E_1. Since f is not constant, we can then proceed as in Case 3.1(b) to obtain the desired result.

3. $E = E_3$
We begin by determining an open set $V^{(2)} \subset V^{(1)}$ within which $f_1 < C < f_2$ for some constant C, and break up E_3 in the following sets:

$$B_1 = \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge f_1(x, y, z) < u < C \wedge v = g(x, y, z, u)\}$$
$$B_2 = \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge C < u < f_2(x, y, z) \wedge v = g(x, y, z, u)\}$$
$$B_3 = \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = C \wedge v = g(x, y, z, u)\}$$

On B_3 we can reason as in Case 3.1(a) for $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset V^{(2)}$, $\pi_{4,5}(B_3(\tau(\blacksquare))) \subseteq \pi_{4,5}(B_3(\tau(\square)))$.

We show how to treat B_1; the treatment of B_2 is analogous. Within a certain open set V to be determined, we are going to break up B_1 in a special way in two overlapping parts of the following form:

$$B_{1,1} = \bigcup_{t \in (0,\delta)} \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = f(x, y, z) + t \wedge v = g(x, y, z, u)\}$$
$$B_{1,2} = \bigcup_{t \in (c_V, C)} \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = t \wedge v = g(x, y, z, u)\}$$

for certain δ and c_V, which we are now going to define.

If f is constant, then $\delta := 0$, and c_V is the constant value of f.

So, suppose that f is not constant in x; the cases y and z are analogous. Then $h_t : (x, y, z) \mapsto (y, z, f(x, y, z) + t)$ is a homeomorphism for every t. Let

$$k : (y, z, u, t) \mapsto g\big((h_t)_x^{-1}(y, z, u), y, z, u\big),$$

where $(h_t)_x^{-1}$ is the function defined by $h_t((h_t)_x^{-1}(y, z, u), y, z) = (y, z, u)$.

We now want to find a δ such that $k_t : (y, z, u) \mapsto k(y, z, u, t)$ is regular for every $t \in (0, \delta)$. Thereto, consider the (semi-algebraic) set

$$D = \left\{(y, z, u, t) \mid (y, z, u) \in h_0\big(V^{(2)}\big) \wedge 0 < t < 1 \wedge \frac{\partial k}{\partial y}(y, z, u, t) = 0\right\}$$

Using a cell decomposition of $\mathbb{R}^4$ compatible with D, we can find an open set $W^{(0)} \subseteq h_0(V^{(2)})$ and a $\delta^{(0)} > 0$ such that on $W^{(0)} \times (0, \delta^{(0)})$ either $\frac{\partial k}{\partial y} = 0$, i.e., k is constant in y, or $\frac{\partial k}{\partial y} \neq 0$, i.e., k is strictly monotone in y. Proceeding similarly, we can find $W^{(2)} \subseteq W^{(1)} \subseteq W^{(0)}$ and $0 < \delta^{(2)} < \delta^{(1)} < \delta^{(0)}$ such that k is either constant or strictly monotone in z on $W^{(1)} \times (0, \delta^{(1)})$, and k is either constant or strictly monotone in u on $W^{(2)} \times (0, \delta^{(2)})$. Hence, within $W^{(2)}$, k_t is regular for every $t \in (0, \delta^{(2)})$.

Next, choose an open set $V^{(3)} \subset V^{(2)}$ and $0 < \delta^{(3)} < \delta^{(2)}$ such that $h_t(V^{(3)}) \subset W^{(2)}$ for every $t \in (0, \delta^{(3)})$. We then restrict $V^{(3)}$ even further to an open set $V^{(4)}$, and simultaneously choose $\delta^{(4)}$ such that the following conditions are satisfied:

$$C - \sup_{V^{(4)}} f > \delta^{(4)} > 0$$
$$\sup_{V^{(4)}} f - \inf_{V^{(4)}} f < \min\left\{\delta^{(3)}, \delta^{(4)}\right\}$$

It is now clear that, within $V := V^{(4)}$, we have $B_1 = B_{1,1} \cup B_{1,2}$ where we put $c_V := \sup_{V^{(5)}} f$ and $\delta := \min\{\delta^{(3)}, \delta^{(4)}\}$.

It remains to deal with $B_{1,1}$ and $B_{1,2}$, but this poses no longer any problems:

$B_{1,1}$: By construction, $k(y, z, u, t)$ is regular for every $t \in (0, \delta)$. This implies that for each $t \in (0, \delta)$ we can treat the set

$$\{(y, z, u, v) \mid (y, z, u) \in h_t(S) \wedge v = k(y, z, u, t)\}$$

as described at the end of Case 3.1(b).

$B_{1,2}$: Here, for every t, we are back in Case 3.1(a).

4. $E = E_4$. We begin again by determining an open set $V^{(2)} \subset V^{(1)}$ within which $f_1 < C < f_2$ for some constant C, and break up E_4 in the following sets:

$$\begin{aligned} B_1 = \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge f_1(x, y, z) < u < C \\ \wedge\, g_1(x, y, z, u) < v < g_2(x, y, z, u)\} \\ B_2 = \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge C < u < f_2(x, y, z) \\ \wedge\, g_1(x, y, z, u) < v < g_2(x, y, z, u)\} \\ B_3 = \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = C \\ \wedge\, g_1(x, y, z, u) < v < g_2(x, y, z, u)\} \end{aligned}$$

On B_3 we can reason as in Case 3.2(a) and for $\tau \in \mathbb{A}$ such that $\tau(\blacksquare) \subset V^{(2)}$, $\pi_{4,5}(B_3(\tau(\blacksquare))) \subseteq \pi_{4,5}(B_3(\tau(\square)))$.

We show how to treat B_1; the treatment of B_2 is analogous. By the same procedure as in Case 4.3 but now working with two functions k_1 and k_2 (one for g_1 and one for g_2), we break up B_1 within a certain open set V:

$$B_{1,1} = \bigcup_{t \in (0,\delta)} \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = f(x, y, z) + t \wedge g_1(x, y, z, u) < v < g_2(x, y, z, u)\}$$

$$B_{1,2} = \bigcup_{t \in (c_V, C)} \{(x, y, z, u, v) \mid (x, y, z) \in S \wedge u = t \wedge g_1(x, y, z, u) < v < g_2(x, y, z, u)\}$$

We finally deal with $B_{1,1}$ and $B_{1,2}$ as follows:

$B_{1,1}$: By construction, $(k_1)_t$ and $(k_2)_t$ are regular for every $t \in (0, \delta)$. Writing $\pi_{4,5}(B_{1,1})$ as

$$\bigcup_{t \in (0,\delta)} \pi_{3,4}(\{(y, z, u, v) \mid [t](y, z, u) \in h_t(S) \wedge k_1(y, z, u, t) < v < k_2(y, z, u, t)\})$$

we can therefore reason analogously as described at the end of Case 3.2(b) for every t individually.

$B_{1,2}$: Here, for every t individually, we are straight back in Case 3.2(a).

The proof of Lemma 4 is complete. □

5 Concluding remarks

We have treated the positive-one pass queries for three-dimensional datasets only. Our proof uses only fairly elementary mathematics. By using more heavy machinery, one can probably prove our Theorem 2 in general for n-dimensional datasets and $n-1$-ary queries. Conceivably this generalisation can also be performed starting from our own proof, but that will be exceedingly laborious.

Extending our proof technique to larger classes of RAEs is not obvious to us. When cartesian product is allowed our basic technique of, given a set Γ, finding an affine transformation τ such that $\tau(S) \cap \Gamma$ is either $\tau(S)$ itself or $\varnothing$ no longer works for $\tau(S) \times \tau(S)$. When negation is allowed, the normal form of Lemma 3 becomes much more complex, with consequences for the case analysis.

Ultimately, one can even go further than the problem posed in the Introduction, and throw in connectivity testing of *parameterized* queries, which can then even be nested [2, 5].

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