An Isotopic Invariant for Planar Drawings of Connected Planar Graphs

J. Hidders

Abstract

In this report we present a data structure called PLA-structure for representing the topological information of a database containing points, lines and areas in a plane. This data structure gives for every point in the database the circular alternating list of incident lines and adjacent areas that one sees as one proceeds clockwise around the point. It will be shown in this report that if we add an indication of which area is the "outside" of the drawing in the plane, the datastructure is a complete isotopic invariant. This means that the data structure contains exactly all the isotopic information of the data. The data structure is therefore interesting as an efficient data structure for spatial data that is mainly queried on its topological aspects.

1 Introduction

The last few years has shown the rise of a new area in database research viz. spatial databases [Par95]. Spatial databases offer next to the conventional database services also the possibilities of storing spatial data and manipulating it with geometric and topological operations. Areas that rely upon these types of databases include CAD-CAM, VLSI, robotics, historical databases, geographical information systems, architectural sciences, visual perception and autonomous navigation, tracking, environmental protection and medical imaging. Typically, the spatial information in these areas is limited to a two-dimensional plane, a sphere or a three-dimensional space. It is one of the main tasks of a spatial database to provide efficient data structures for this kind of information.

Whether a data structure is efficient or not is dependent upon the type of queries and updates that is going to be performed. It is, for instance, not necessary for all queries to know the exact coordinates or the exact shape of the queried spatial objects. For instance, the query "Give all cities in Germany on the west bank of the Rhine north of Bonn" only requires knowledge of what regions objects are contained in and how they are ordered in longitude and latitude. The query "Is there an airport within 100 miles of my house?" only needs knowledge of distances between objects. The data structure we will be discussing here is meant for queries that use only the topological properties of the objects. These are properties such as adjacency, connectivity and containment, that remain invariant under topological deformation. Examples of such queries are "What states of the US are bordering on Ohio?" and "Is there a highway connecting Tampa with Miami?". Such queries can be computed more efficiently if the data base stores the topological relationships between the objects explicitly.

A common representation of point-line-area spatial databases is by a data structure listing for each point its incident lines and its adjacent areas, arranged in the order in which they appear as one proceeds clockwise around the point. We call this data structure an *observation-structure*. Essentially this structure underlies the TIGRIS system [Her87], as well as the topological layer of the ARC/INFO system [Mor89], and the original design of the cartography system of the Census Bureau of the United States [Cor79]. We intend to show in this report that this data structure with a slight extension caputures exactly *all* topological information of a point-line-area database, i.e., two datastructure are equivalent iff the spatial databases they describe are topologically equivalent.

2 Spatial Databases and Observation Structures

In this section, we define what we consider a spatial database for the discussion in this report and how it is described by an observation structure.

Definition 1 *A* spatial database *consists of a finite set of named points, a finite set of named lines and a finite set of named areas. Each point name is assigned to a distinct point in the plane. Each line name is assigned to a distinct non-selfintersecting continuous curve in the plane that starts and ends in a named point and does not contain any other named points except these. Each area name is assigned to a distinct area formed by the named lines.*

A spatial database corresponds with the drawing of an undirected multigraph with nodes, edges and regions enclosed by edges uniquely labeled, and no intersecting edges. Note that lines are allowed to start and end in the same point, i.e., the database may contain loops. An example of a spatial database is given in Figure 1. Here we see a database with 6 points, 10 edges and 6 areas. Note that line names are Roman captals, point names are Roman characters and area names are Greek characters.

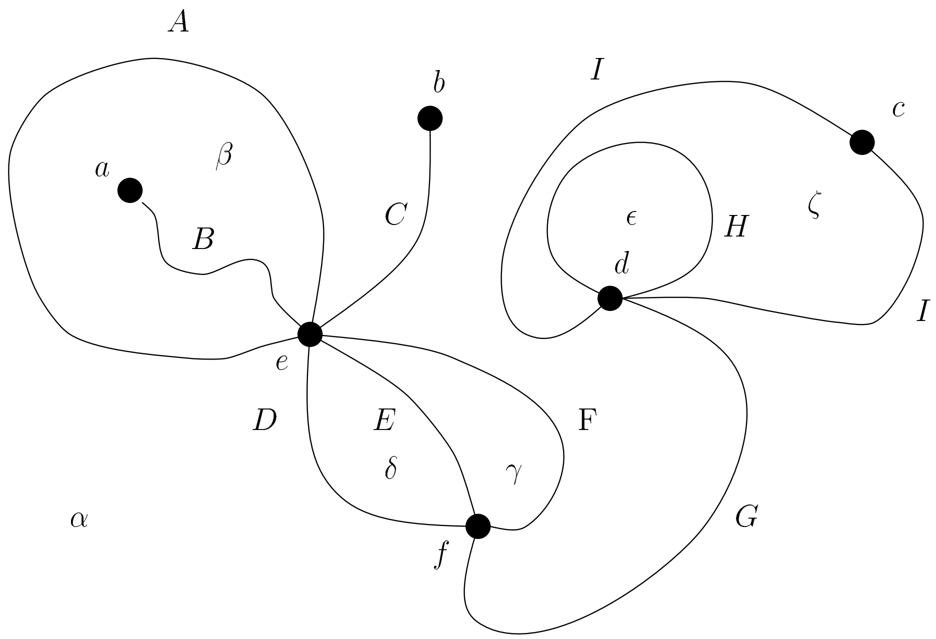

Figure 1: An example of a spatial database

Definition 2 *An* observation of a point in a spatial database *is a circular alternating list of area names and line names corresponding respectively to the areas and lines that an observer, placed in the named point, sees when he makes one clockwise full turn and scans the environment of the point.*

This last definition is rather informal but it is shown in [KPV95] that it is a well-defined notion. This is based upon the following observation. If we draw a circle of diameter d around the point then we can make a circular alternating list of area names and line names corresponding to areas and lines that we meet when we follow the circle clockwise. It can be proved that if we make d small enough all circles with a smaller diameter will have the same circular list of area names and line names. It is this unique circular list that is defined as the observation of that point.

An example of an observation is that of point f in Figure 1 whose alternating list is $(\alpha D \delta E \gamma F \alpha G)$. We are now ready to define the data structure that has to contain all the topological information of a spatial database.

Definition 3 *A* PLA-structure of a spatial database D *is defined as a tuple* $\langle P, L, A, \alpha^\infty, Obs \rangle$ *where* P *is the set of point names of* D*,* L *is the set of line names of* D*,* A *is the set of area names of* D*,* α^∞ *is the name of the unbounded area of* D*, and* Obs *is a function that maps every point name in* P *to the observation of the point with that name in* D*.*

It has been shown in [KPV95] that there is an efficient algorithm for deciding whether an arbitrary PLA-structure actually represents a possible spatial database.

3 The Invariance of PLA-Structures

In this section we will prove that PLA-structures contain exactly all the topological information of a spatial database. In order to define this more formally we need the notion of "topological equivalence". An example of two topological equivalent spatial databases is given in Figure 2. Intuitively, two spatial databases are topologically equivalent if one can be obtained form the other by a continuous deformation. In other words, there is a "continuous motion picture" in the plane by which one is transformed into the other. The mathematical formalization of such "a motion picture" is given by the notion of *isotopy*. An isotopy h is a continuous series $(h_t \mid 0 \leq t \leq 1)$ of homeomorphisms of the plane.

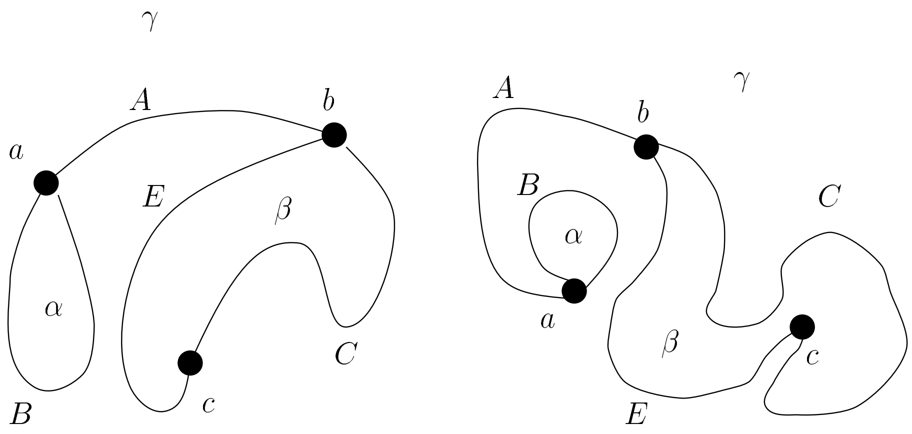

Figure 2: Two topologically equivalent spatial databases

Definition 4 *Two spatial databases D_1 and D_2 are called* topological equivalent *if there exists an isotopy h such that $h_0(D_1) = D_1$ and $h_1(D_1) = D_2$, with the understanding that h respects the names of points, lines and areas.*

A data structure contains exactly all topological information of a spatial databases if two data structures are equal iff they represent two topological equivalent databases. It is the central theorem of this report that this holds for PLA-structures.

Theorem 1 *Two PLA-structures of two spatial databases are equal iff the two spatial databases are topological equivalent.*

It can now be explained why PLA-structures contain an indication of the unbounded region next to the observations of all the points in the database. In Figure 3 we see two spatial databases with the same observations for all the points but who are not isotopically equivalent.

The proof of Theorem 1 is done in several stages. We will begin with the the if-part.

Lemma 2 *The PLA-structures of two topological equivalent databases are equal.*

Proof: It is trivial to see that the names of points, lines and areas remain the same since every isotopy defining equivalency respects them by definition. It also holds that the isotopy lets bounded areas remain bounded.

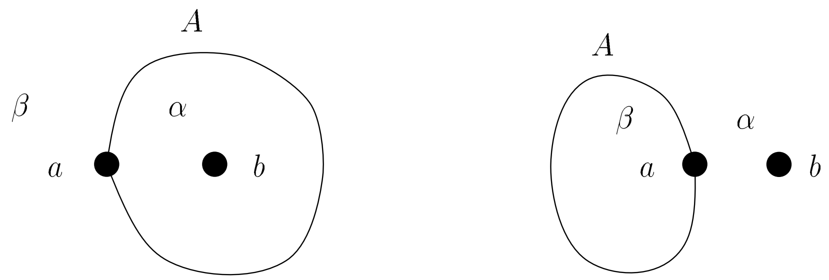

Figure 3: Two spatial databases with the same observations

Therefore the (unique) unbounded area also remains unbounded. Finally, the observation of every point is not changed by the topology because a circle with center point p, that is located inside a toplogical deformation of a (possibly other) circle that was used to obtain an observation from p, gives rise to an identical observation as the original circle. □

The proof for the only-if-part of the theorem is given first only for "connected" spatial databases. A spatial database is connected if the graph defined by its points and lines is connected.

Lemma 3 *If two connected spatial databases have the same PLA-structure then they are isotopically equivalent.*

Proof: Every connected spatial database can be built using the following three steps: (1) adding a point to an empty database, (2) adding a point and a line connecting it to an already existing point, and (3) adding a line between two already existing points. Note that the third step causes an area to be split into two areas. The result of these steps is always again a connected spatial database. We will prove that if the same step is performed upon two topologically equivalent spatial databases with the same PLA-structure then the results will also be topologically equivalent provided that their new PLA-structures are the same. From this it follows with induction upon the number of steps that if we construct two connected spatial databases with the same PLA-structure using the same list of steps, we will end up with two topologically equivalent spatial databases. Since any two connected spatial databases with the same PLA-structure can be constructed using the same list of steps it follows that they are topologically equivalent.

In the following of the proof we will use D_1 and D_2 for the two topologically equivalent spatial databases and h for the isotopy that makes them equivalent. Furthermore, we use $\langle P, L, A, \alpha^\infty, Obj\rangle$ for the PLA-structure of D_1 and D_2. The databases after the step are called D'_1 and D'_2 and their common PLA-structure is $\langle P', L', A', \beta^\infty, Obj'\rangle$.

Adding a point to an empty database It is trivial to see that two spatial databases with one point and the same PLA-structure are topologically equivalent.

Adding a point and a line connecting it to an old point In both D_1 and D_2 the new point and line are placed in the same region. The isotopy h maps this region of D_1 containing the new line, to the region with the same name in D_2. Therefore we can obtain an isotopy between D'_1 and D'_2 by extending h with a transformation changing only points in this region such that the new line and new point are moved from their position in $h_1(D'_1)$ to their position in D'_2.

Adding a line between two points If the new line lies in a bounded region then we can use the same technique as before; the old isotopy will keep this line in the same region and h can be extended to move the line to its place in D'_2. If, however, the new line lies in the α^∞ it is slightly more complicated. The new line will split α^∞ in a bounded area γ and an unbounded area β^∞. If we look at the observations of the two points involved we see that if one point sees β^∞ left from the new line then the other point will see it at the right and vice versa. Which of the two is the case is determined by the PLA-structure of D'_1 and D'_2. So, $h_1(D_1)$ is identical to D'_2 except for the new line splitting α^∞ in γ and β^∞. Since the points that this line is connected to, see in both spatial databases γ at the same side of the line, the topology h can be extended with a transformation changing only points in α^∞ that moves the new line to its position in D'_2.

□

The lemma for connected spatial databases can be easily generalized for all spatial databases.

Lemma 4 *If two spatial databases have the same PLA-structure then they are isotopically equivalent.*

Proof: Every spatial database can be regarded as a set of connected spatial databases with recursively sets of spatial databases nested in their bounded regions. Two spatial databases with the same PLA-structure will always have the same nesting depth. We can prove by induction upon the depth of nesting that the theorem holds. The case where there is no nesting follows directly from Lemma 3. If we have a spatial database with depth of nesting $n + 1$ then we first omit the nested spatial databases and then construct an isotopy by Lemma 3 between the remaining connected spatial databases. This isotopy leaves the nested spatial databases in the "right" area. Therefore we only need to extend the isotopy for every area with nested spatial databases with a transformation that moves only the points in this region, to move these nested spatial databases to their right place and form within this region. That this is possible follows from the induction hypthesis and the fact that the nesting depth of these nested spatial databases is less than or equal to n. $\square$

It is now simple to see how the proof of Theorem 1 follows from Lemma 2 and Lemma 4. This concludes the proof of the central theorem of this report.

4 Conclusions

We have shown that the PLA-structure is a data structures that captures exactly all the topological information of a spatial database containing points, lines and areas in a plane. This identifies it is an important candidate for an efficient data structure for spatial data that is mainly queried on its topological content. It also shows that it might serve as an appropriata conceptual model for users that are mainly interested in topological properties of their spatial data. These users can then be sure that their conceptual model contains *all* the topological aspects of their data.

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