

# First-Order Complete and Computationally Complete Query Languages for Spatio-Temporal Databases

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We address a fundamental question concerning spatio-temporal database systems: “What are exactly spatio-temporal queries?” We define spatio-temporal queries to be computable mappings that are also *generic*, meaning that the result of a query may only depend to a limited extent on the actual internal representation of the spatio-temporal data. Genericity is defined as invariance under groups of geometric transformations that preserve certain characteristics of spatio-temporal data (e.g., collinearity, distance, velocity, acceleration, . . .). These groups depend on the notions that are relevant in particular spatio-temporal database applications. These transformations also have the distinctive property that they respect the monotone and unidirectional nature of time.

We investigate different genericity classes with respect to the constraint database model for spatio-temporal databases and we identify sound and complete languages for the first-order and the computable queries in these genericity classes. We distinguish between genericity determined by time-invariant transformations, genericity notions concerning physical quantities and genericity determined by time-dependent transformations.

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## 1. INTRODUCTION

Since the early 1990s, various database systems have been developed to handle spatial data [Abel and Ooi 1993; Buchmann 1989; Egenhofer and Herring 1995; Gunther and Schek 1991; Güting 1999; Rigaux et al. 2000; Scholl and Voisard 1997] and solid theories for such systems have been proposed and studied [Paredaens et al. 1994; Kuper and Scholl 2000]. Conceptually, spatial databases contain possibly infinite sets of points in a real space  $\mathbf{R}^n$ . In more recent years, we have seen the emergence of database systems and applications that are dealing with *spatio-temporal data* [Böhlen et al. 1999; Chen and Zaniolo 2000; Frank et al. 1999; Güting et al. 2000; Pfoser and Tryfona 1998]. Conceptually, spatio-temporal data can be modeled as infinite spatial sets that move or change in time, that is, sets in  $\mathbf{R}^n \times \mathbf{R}$ .

We can distinguish between three main approaches towards modeling spatio-temporal data: one based on new data types, the second focusses on moving objects and the third is based on the constraint database model.

In the data type approach [Frank et al. 1999], a set of base, spatial, temporal and spatio-temporal data types is proposed. The (two-dimensional) spatial data types are “point”, “points” (a finite set of points), “line”, and “region”. Time is to be considered linear and continuous. A type constructor named “moving” takes any type  $\alpha$  and provided a mapping from time to  $\alpha$ . In this way, moving types can be modeled. On top of this, a set of spatial operators are defined which can be lifted to spatio-temporal operators. For example, the intersection operator defined on “region” and a “point” can be lifted such that it computes the intersection of moving regions and points. Those operators are embedded in SQL-like languages and the geometry is most often considered to be discrete (polylines, polygons, ...).

One particular line of research in the field of spatio-temporal databases, started by Wolfson, concentrates on *moving object databases* (MODs) [Güting and Schneider 2005; Wolfson 2002], a field in which several data models and query languages have been proposed to deal with moving objects whose position is recorded at, not always regular, moments in time. Some of these models are geared towards handling uncertainty that may come from various sources (measurements of locations, interpolation, ...) and often ad-hoc query formalisms

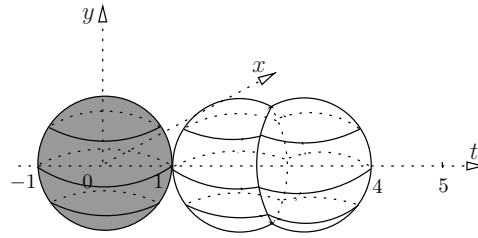


Fig. 1. An example of a spatio-temporal database in  $\mathbf{R}^2 \times \mathbf{R}$ .

have been proposed [Su et al. 2001]. For an overview of models and techniques in this field, we refer to the recent textbook by Güting and Schneider [2005].

The third approach, the constraint approach, considers the data to be continuous. It is a recent and much acclaimed method for effectively representing infinite geometrical figures is provided by the *constraint database model*, that was introduced in Kanellakis et al. [1990, 1995] (an overview of the area of constraint databases appeared in Paredaens et al. [2000]; and Revesz [2002] wrote a textbook on this topic).

Until recently, this model has been used mainly in the area of spatial databases, but it provides an equally elegant and efficient way to model spatio-temporal data [Chen and Zaniolo 2000; Chomicki and Revesz 1997, 1999; Grumbach et al. 1998; Kuijpers et al. 2000]. In the setting of the constraint model, a spatio-temporal relation in  $\mathbf{R}^n \times \mathbf{R}$  is finitely represented as a Boolean combination of polynomial equalities and inequalities. Figure 1 depicts the spatio-temporal database  $\{(x, y, t) \mid x^2 + y^2 + t^2 \leq 1 \vee (x^2 + y^2 + (t - 2)^2 = 1 \wedge t \leq 5/2) \vee (x^2 + y^2 + (t - 3)^2 = 1 \wedge t > 5/2)\}$  in  $\mathbf{R}^2 \times \mathbf{R}$ . A spatio-temporal database is a finite collection of such relations and can be finitely represented by the polynomial constraint formulas that represent its relations.

A number of theoretical studies have appeared on the status of time and its relation with space in systems that model moving objects. Erwig et al. [1999] give a taxonomy of applications ranging from those that rely on a step-wise constant geometry to applications which need more complete integration of space and time (like for instance a continuous description of a trajectory). MOST [Sistla et al. 1997], an example of the latter category, relies on a strong interaction of the space and time components (since the space variables are described by linear polynomials in time) and provides a query language that is a combination of a spatial query language and a temporal logic. On the other range of the spectrum, variable independence (defined in terms of orthographic dimension) gives rise to a less expressive data model which has the advantage of a lower complexity of query evaluation [Grumbach et al. 1998; Libkin 2000].

We study spatio-temporal queries from the perspective of expressive power, and do this against the background of the full modeling and querying power of the constraint database model and the first-order and computationally complete languages it offers. We ask which expressions in these languages may be considered as *reasonable* spatio-temporal queries. In database theory, it is usually required that the result of queries should only to a certain limited extent depend on the actual internal representation of databases and that queries

should only ask for properties that are shared by “isomorphic” encodings of the same data. The meaning of “isomorphic” may be influenced by the actual database application and by which notions are relevant to it. In the context of the relational database model, Chandra and Harel [1980] formalized this independence of the actual encoding in terms of the notion of *genericity*. Paredaens et al. [1994] identified a hierarchy of genericity classes for spatial database applications. The generic queries in the different classes focus on different geometrical and topological aspects of the spatial data. On a technical level, generic queries are defined as being invariant under those transformations of the data that preserve the relevant aspects of the data. Whereas Chandra and Harel [1980] considered the group of the isomorphisms (that possibly fix some elements of the domain) in the case of relational databases, Paredaens et al. [1994] identified different geometrical and topological transformation groups (affinities, isometries, translations, homeomorphisms, . . .) for spatial database applications.

We define spatio-temporal queries to be computable mappings that are also *generic*, meaning that the result of a query may only depend to a limited extent on the actual internal representation of the spatio-temporal data. Genericity is defined as invariance under some (application-dependent) group of geometric transformations. These transformations preserve certain characteristics of spatio-temporal data (e.g., collinearity, distance, velocity, acceleration, . . .).

We investigate which notions of genericity are appropriate for spatio-temporal databases and which transformation groups express them. We observe that the transformations should first and foremost respect the monotone and unidirectional nature of time, that is, leave the temporal order of events unchanged. It follows that the relevant transformation groups are the product of a group of time-(in)dependent spatial transformations and a group of monotone increasing transformations of the time-component of the spatio-temporal data. Next, we focus on the former groups and study which of them leave different spatial and spatio-temporal properties (like collinearity, distance and orientation) unchanged. We also focus on physical properties of spatio-temporal data (like velocity and acceleration). The transformation groups that we consider are all subgroups of the time-dependent or time-independent affinities of  $\mathbf{R}^n \times \mathbf{R}$ .

We study the notion of spatio-temporal genericity relative to two popular query languages in the constraint model: first-order logic over the reals (FO) and an extension of this logic with a while-loop (FO + while). Both languages are known to be effectively computable (given termination in the case of FO + while-programs) and FO + while is known to be a computationally complete language on spatio-temporal databases [Van den Bussche 2000]. First, we show that all the genericity classes are undecidable. We show that the considered classes of generic first-order queries are recursively enumerable, however. Hereto, we define first-order point-based languages in which variables are assumed to range over points in  $\mathbf{R}^n \times \mathbf{R}$  and which contain certain point predicates (such as **Between** and **Before**). These point-based languages are shown to be sound and complete for the first-order queries in the considered genericity classes. We have also shown that extensions of these point-based logics with a while-loop give sound and complete languages for the computable queries in the different genericity classes. Our results are inspired by similar

results that were obtained by Gyssens et al. [1999] in the context of spatial databases. Also, the proof techniques we use for time-independent transformation groups, are generalisations of techniques introduced in those papers. However, our results for genericity notions described by time-dependent transformations require new proof techniques.

This article is organized as follows: In Section 2, we define spatio-temporal databases, spatio-temporal queries, and the constraint query languages FO and FO + while. In Section 3, we define a number of genericity notions. In Section 4, we present sound and complete first-order query languages for the different notions of genericity. In Section 5, we present sound and complete languages for the computable queries satisfying the different notions of genericity. We end with a discussion in Section 6.

## 2. DEFINITIONS AND PRELIMINARIES

We denote the set of the real numbers by  $\mathbf{R}$  and the  $n$ -dimensional real space by  $\mathbf{R}^n$ .

Throughout this article, we use the following notational convention. Variables that range over real numbers are denoted by characters  $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \dots$ . When there is the need to distinguish between real variables that indicate spatial coordinates and time coordinates, we use  $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \dots$  for the former and use  $t, t_1, t_2, \dots$  for the latter. Variables that range over vectors in  $\mathbf{R}^n$  and that represent spatial information are denoted by bold characters  $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \dots$ . Real constants are represented by characters  $a, b, c, a_1, b_1, c_1, a_2, b_2, c_2, \dots$ . When there is the need to distinguish between real constants that indicate spatial coordinates and time coordinates, we use  $a, b, c, a_1, b_1, c_1, a_2, b_2, c_2, \dots$  for the former and use Greek characters  $\tau, \tau_1, \tau_2, \dots$  for the latter.

Finally, bold characters  $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \dots$  represent constant  $n$ -dimensional spatial vectors.

Vectors  $(\mathbf{a}, \tau)$  containing mixed spatial and temporal information are denoted  $p, q, r, p_1, q_1, r_1, p_2, q_2, r_2, \dots$  and variable vectors  $(\mathbf{x}, t)$  are represented by characters  $u, v, w, u_1, v_1, w_1, u_2, v_2, w_2, \dots$ .

### 2.1 Semi-Algebraic and Spatio-Temporal Databases

We consider  $n$ -dimensional spatial figures that move or change over time. A moving figure in  $\mathbf{R}^n$  can be described by means of an infinite set of tuples  $(a_1, a_2, \dots, a_n, \tau)$  in  $\mathbf{R}^n \times \mathbf{R}$ , where  $(a_1, a_2, \dots, a_n)$  represent the  $n$ -dimensional spatial coordinates of  $(a_1, a_2, \dots, a_n, \tau)$  and  $\tau$  its time coordinate. Obviously, this infinite information needs to be represented finitely in order to be stored in the memory of a computer. In this section, we describe two approaches to model such changing figures, namely the *semi-algebraic database model* and the *spatio-temporal database model*. Semi-algebraic databases are based on real numbers, while spatio-temporal databases are based on  $(n + 1)$ -dimensional points. Both models resort under the constraint database model [Paredaens et al. 2000; Revesz 2002]. Neither of these models require the explicit storage of real numbers.

*Definition 2.1.* A *semi-algebraic relation in  $\mathbf{R}^n$*  is a subset of  $\mathbf{R}^n$  that can be described as a Boolean combination of sets of the form

$$\{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid p(x_1, x_2, \dots, x_n) > 0\},$$

with  $p$  a polynomial with integer coefficients in the real variables  $x_1, x_2, \dots, x_n$ .

In mathematical terms, semi-algebraic relations are known as *semi-algebraic sets* [Bochnak et al. 1987]. In this article, we will be mainly interested in semi-algebraic relations in real spaces of the form  $\mathbf{R}^{(n+1) \times k}$ . These relations can be viewed as  $k$ -ary relations over  $\mathbf{R}^n \times \mathbf{R}$  (i.e., the  $n$ -dimensional space extended with a time dimension). The next example illustrates this for  $k = 1$  and  $n = 2$ .

*Example 2.2.* Figure 1 gives an example of a semi-algebraic relation in  $\mathbf{R}^3$ . This set can be described as follows:  $\{(x, y, t) \in \mathbf{R}^2 \times \mathbf{R} \mid x^2 + y^2 + t^2 \leq 1 \vee (x^2 + y^2 + (t - 2)^2 = 1 \wedge t \leq 5/2) \vee (x^2 + y^2 + (t - 3)^2 = 1 \wedge t > 5/2)\}$ . This relation shows at its beginning, that is, at  $t = -1$ , a single point in the origin of  $\mathbf{R}^2$ . Then, it shows a disk whose radius increases and later decreases and ends in a point at moment  $t = 1$ , followed by a circle whose radius increases, decreases, increases and then shrinks to a point.

We call a semi-algebraic relation in  $\mathbf{R}^n$  also a *semi-algebraic relation of arity  $n$* . A semi-algebraic database is essentially a finite collection of semi-algebraic relations. We define this now.

*Definition 2.3.* A (*semi-algebraic*) *database schema  $\sigma$*  is a finite set of relation names, where each relation name  $R$  has a natural number  $ar(R)$ , called its *arity*, associated to it.

Let  $\sigma$  be a database schema. A *semi-algebraic database over  $\sigma$*  is a structure  $\mathcal{D}$  over  $\sigma$  with domain  $\mathbf{R}$  such that, for each relation name  $R$  of  $\sigma$ , the associated relation  $R^{\mathcal{D}}$  in  $\mathcal{D}$  is a semi-algebraic relation of arity  $ar(R)$ .

*Example 2.4.* Let  $\sigma = \{R, S\}$ , with  $ar(R) = 2$  and  $ar(S) = 1$  be a semi-algebraic database schema. Then the structure  $\mathcal{D}$  given by

$$(\mathbf{R}, R^{\mathcal{D}} = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}, S^{\mathcal{D}} = \{x \in \mathbf{R} \mid 0 \leq x \leq 1\})$$

is an example of a semi-algebraic database over  $\sigma$  that contains the open unit disk and the closed unit interval.

We now define spatio-temporal databases. In contrast to semi-algebraic databases, in which points are described by their real coordinates, spatio-temporal databases are based on  $(n + 1)$ -dimensional points. The domain of a spatio-temporal database is  $\mathbf{R}^n \times \mathbf{R}$ . We prefer the notation  $\mathbf{R}^n \times \mathbf{R}$  over  $\mathbf{R}^{n+1}$  for the domain because it stresses the distinction between the time coordinate and the  $n$  spatial coordinates of the  $(n + 1)$ -dimensional points.

In the following definition, we work with  $\mathbf{R}^n \times \mathbf{R}$  as the domain of the spatio-temporal databases and we assume that this *underlying dimension  $n$*  is fixed on before hand. In this article, we assume, for technical reasons that will become clear in Section 4, that  $n \geq 2$ .



Throughout this article we will often use the canonical bijection

$$can : (\mathbf{R}^n \times \mathbf{R})^k \rightarrow \mathbf{R}^{(n+1) \times k}$$

that maps a tuple  $((\mathbf{a}_1, \tau_1), \dots, (\mathbf{a}_k, \tau_k))$  to  $(a_{1,1}, \dots, a_{1,n}, \tau_1, \dots, a_{k,1}, \dots, a_{k,n}, \tau_k)$ , where for  $1 \leq i \leq k$  and  $1 \leq j \leq n$ ,  $a_{i,j}$  denotes the  $j$ th real coordinate of the vector  $\mathbf{a}_i$ , and where  $\tau_i$  denotes the time coordinate of the vector  $\mathbf{a}_i$ .

*Definition 2.5.* A (spatio-temporal) database schema  $\sigma$  is a finite set of relation names, where each relation name  $R$  has a natural number  $ar(R)$ , called its arity, associated to it.

A subset of  $(\mathbf{R}^n \times \mathbf{R})^k$  is a *spatio-temporal relation of arity  $k$*  if its image under the canonical bijection  $can : (\mathbf{R}^n \times \mathbf{R})^k \rightarrow \mathbf{R}^{(n+1) \times k}$  is a semi-algebraic relation of arity  $(n+1) \times k$ .

Let  $\sigma$  be a spatio-temporal schema. A *spatio-temporal database over  $\sigma$*  is a structure  $\mathcal{F}$  over  $\sigma$  with domain  $\mathbf{R}^n \times \mathbf{R}$  such that to each relation name  $R$  in  $\sigma$ , a spatio-temporal relation  $R^{\mathcal{F}}$  of arity  $ar(R)$  is associated to it.

*Remark 2.6.* A spatio-temporal database  $\mathcal{F}$  over  $\sigma$  can be viewed in a natural way as a semi-algebraic database  $\overline{\mathcal{F}}$  over the semi-algebraic schema  $\overline{\sigma}$ , which has for each relation name  $R$  of  $\sigma$ , a relation name  $\overline{R}$  of arity  $(n+1) \times ar(R)$ . For each relation name  $R$ ,  $\overline{R}^{\overline{\mathcal{F}}}$  is obtained from  $R^{\mathcal{F}}$  by applying the canonical bijection  $can : (\mathbf{R}^n \times \mathbf{R})^{ar(R)} \rightarrow \mathbf{R}^{(n+1) \times ar(R)}$ . We will use the notation introduced here throughout the article.

Following this remark, we observe that spatio-temporal relations and databases can be *finitely* encoded and stored by means of the systems of polynomial equalities and inequalities (i.e., by means of a quantifier-free formula of first-order logic over the reals with  $+$ ,  $\times$ ,  $<$  and the constants 0 and 1) that describe the associated semi-algebraic relations and databases.

*Remark 2.7.* Throughout this article, we assume that databases are finitely encoded by systems of polynomial equations and that a specific data structure is fixed (possible data structures are dense or sparse representations of polynomials). The specific choice of data structure is not relevant to the topic of this article, but we assume that one is fixed. When we talk about computable queries later on, we mean Turing computable with respect to the chosen encoding and data structures.

The model presented here and the results in this article can be extended straightforwardly to the situation where spatio-temporal relations are accompanied by classical thematic information, like the typical alpha-numeric data you find in business applications and also, in combination with spatial data, in Geographical Information Systems. However, because the problem that is discussed here is captured by this simplified model, we stick to it for reasons of simplicity of exposition.

*Example 2.8.* Figure 1 in the Introduction gives an illustration of a spatio-temporal database over a schema  $\sigma = \{R\}$  with underlying dimension 2, where  $R$  has arity 1.

*Definition 2.9.* Let  $\sigma$  be a spatio-temporal schema and let  $\mathcal{F}$  be a spatio-temporal database over  $\sigma$  with underlying dimension  $n$ . Let  $R$  be a relation name in  $\sigma$  and let  $\tau_0$  be a real number representing a moment in time. We call the subset

$$R^{\mathcal{F}} \cap (\mathbf{R}^n \times \{\tau_0\})^{ar(R)}$$

of  $(\mathbf{R}^n \times \{\tau_0\})^{ar(R)}$  the *snapshot of  $R$  at the moment  $\tau_0$* . The snapshot of the spatio-temporal database  $\mathcal{F}$  at the moment  $\tau_0$  is the finite set of snapshots of all its relations at  $\tau_0$ .

*Example 2.10.* For the spatio-temporal relation depicted in Figure 1, the snapshot at  $-1$  is  $\{(0, 0, -1)\}$ , the snapshot at  $0$  is the closed unit disk in the plane  $t = 0$  and the snapshot at  $5$  is the empty set.

## 2.2 Spatio-Temporal and Semi-Algebraic Database Queries

Here, we define spatio-temporal and semi-algebraic database queries as computable mappings of some type. In the next section, we will argue that not all such mappings should be regarded as “reasonable” queries and that further conditions on the nature of these mappings have to be imposed.

*Definition 2.11.* Let  $\sigma$  be a semi-algebraic database schema. A  *$k$ -ary semi-algebraic database query  $Q$  over  $\sigma$*  is a partial, computable mapping (in the sense of Remark 2.7) from the set of semi-algebraic databases over  $\sigma$  to the set of  $k$ -ary semi-algebraic relations.

*Definition 2.12.* Let  $\sigma$  be a spatio-temporal database schema and let us consider input spatio-temporal databases over  $\sigma$  with underlying dimension  $n$ . A  *$k$ -ary  $n$ -dimensional spatio-temporal database query  $Q$  over  $\sigma$*  is a partial, computable mapping (in the sense of Remark 2.7) from the set of spatio-temporal databases over  $\sigma$  to the set of  $k$ -ary spatio-temporal relations with underlying dimension  $n$ .

We also call a  $k$ -ary  $n$ -dimensional spatio-temporal database query a *spatio-temporal database query of output type  $(n, k)$* .

Note that we restrict spatio-temporal database queries to preserve the underlying dimension of the input database.

*Example 2.13.* Let  $\sigma = \{R\}$ , where  $R$  has arity 1 and let the underlying dimension be 2. The query that selects those snapshots from the relation  $R$  where  $R$  shows a circle is a spatio-temporal database query of output type  $(2, 1)$ . Applied to the database of Example 2.8 and shown in Figure 1, this query returns the union of its snapshots in the open time interval  $]1, 4[$ .

There is a natural way to see spatio-temporal queries as semi-algebraic queries, that is captured in the following definition of equivalence of queries.

*Definition 2.14.* Let  $\sigma$  be a spatio-temporal database schema and let us consider input spatio-temporal databases over  $\sigma$  with underlying dimension  $n$ . Let  $\bar{\sigma}$  be the corresponding semi-algebraic database schema (see Remark 2.6). Let  $Q$  be a  $k$ -ary  $n$ -dimensional spatio-temporal database query over  $\sigma$  and let



$\overline{Q}$  be a  $((n + 1) \times k)$ -ary semi-algebraic database query over  $\overline{\sigma}$ . We say that  $Q$  and  $\overline{Q}$  are *equivalent* if for every database  $\mathcal{F}$  over  $\sigma$  we have

$$\overline{Q(\mathcal{F})} = \overline{Q}(\overline{\mathcal{F}}).$$

### 2.3 First-Order Logic and Its Extension with a While Loop as a Spatio-Temporal Query Language

First-order logic over the field of the real numbers,  $\text{FO}(+, \times, <, 0, 1)$  for short, has been well-studied as a query language for spatial databases [Kanellakis et al. 1995; Paredaens et al. 1994, 2000]. In the setting of spatio-temporal databases, it can be used as a query language in a similar way. We introduce  $\text{FO}(+, \times, <, 0, 1)$  here as a spatio-temporal query language, built on semi-algebraic databases that represent spatio-temporal databases.

*Definition 2.15.* Let  $\sigma = \{R_1, R_2, \dots, R_m\}$  be a spatio-temporal database schema and let us consider queries working on input databases over  $\sigma$  with underlying dimension  $n$ . Let  $\overline{R}_i$  ( $1 \leq i \leq m$ ) be the corresponding semi-algebraic relation names of arity  $(n + 1) \times \text{ar}(R_i)$  (we follow the notation of Remark 2.6) and let  $\overline{\sigma}$  be the semi-algebraic schema  $\{\overline{R}_1, \overline{R}_2, \dots, \overline{R}_m\}$ .

Let  $\varphi(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2, \dots, \mathbf{x}_k, t_k)$ , be a first-order logic formula over the alphabet  $(+, \times, 0, 1, <, \overline{R}_1, \overline{R}_2, \dots, \overline{R}_m)$ . If  $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n})$ , then the free variables of  $\varphi$  are  $x_{1,1}, \dots, x_{1,n}, t_1, x_{2,1}, \dots, x_{2,n}, t_2, \dots, x_{k,1}, \dots, x_{k,n}, t_k$ . The formula  $\varphi$  expresses a semi-algebraic  $((n + 1) \times k)$ -ary query  $\overline{Q}$  which is equivalent to a  $k$ -ary  $n$ -dimensional spatio-temporal query  $Q$ . For each input spatio-temporal database  $\mathcal{F}$  over  $\sigma$ ,  $Q(\mathcal{F})$  is defined as the set of points  $((\mathbf{a}_1, \tau_1), (\mathbf{a}_2, \tau_2), \dots, (\mathbf{a}_k, \tau_k))$  of  $(\mathbf{R}^n \times \mathbf{R})^k$  such that

$$(\mathbf{R}, +, \times, 0, 1, <, \overline{R}_1^{\overline{\mathcal{F}}}, \overline{R}_2^{\overline{\mathcal{F}}}, \dots, \overline{R}_m^{\overline{\mathcal{F}}}) \models \varphi[\mathbf{a}_1, \tau_1, \mathbf{a}_2, \tau_2, \dots, \mathbf{a}_k, \tau_k],$$

where  $\varphi[\mathbf{a}_1, \tau_1, \mathbf{a}_2, \tau_2, \dots, \mathbf{a}_k, \tau_k]$  denotes the formula  $\varphi(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2, \dots, \mathbf{x}_k, t_k)$  with its free variables instantiated by  $\mathbf{a}_1, \tau_1, \mathbf{a}_2, \tau_2, \dots, \mathbf{a}_k, \tau_k$ .

We will refer to the first-order query language, introduced here, as  $\text{FO}(+, \times, <, 0, 1, \overline{R}_1, \overline{R}_2, \dots, \overline{R}_m)$ , or, if the schema is clear from the context, as  $\text{FO}(+, \times, <, 0, 1)$ .

*Example 2.16.* As in Example 2.13, let  $\sigma = \{R\}$ , where  $R$  has arity 1 and let the underlying dimension be 2. The formula  $(\exists x_0)(\exists y_0)(\exists r > 0)((x - x_0)^2 + (y - y_0)^2 = r^2 \leftrightarrow \overline{R}(x, y, t))$  expresses a spatio-temporal query of output type  $(2, 1)$ . It selects those snapshots from a spatio-temporal relation  $R$  where  $R$  shows a circle. As mentioned, applied to the database of Example 2.8, this query returns all its snapshots in the time interval ]1, 4[.

We remark that the formalism of semi-algebraic and spatio-temporal databases and the first-order query language introduced here, fits within the framework of *constraint databases* [Paredaens et al. 2000; Revesz 2002]. It is well known that  $\text{FO}(+, \times, <, 0, 1)$ -formulas can be effectively evaluated in the constraint model and therefore also in this context. It is also known that the output can be represented in the same constraint formalism [Paredaens et al. 2000; Van den Bussche 2000], that is, by a quantifier-free formula over  $(+, \times, 0, 1, <)$ .

*Remark 2.17.* Although any FO-formula expresses a database query, it does not necessarily express a spatio-temporal database query. Indeed, a spatio-temporal database query should always return a spatio-temporal relation (see Definition 2.12). The following example shows that not every FO-formula guarantees this: The formula

$$(\exists t)\overline{R}(x_1, x_2, t)$$

expresses the projection of the spatio-temporal relation  $R$  on the spatial  $(x_1, x_2)$ -plane. The formula returns a set of couples  $(x_1, x_2)$  in  $\mathbf{R}^2$  that form a semi-algebraic set with a purely spatial meaning. In other words, this formula does not define a spatio-temporal relation.

We end this section by specifying the programming language  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma}) + \text{while}$  which is known to be computationally complete on semi-algebraic databases [Gyssens et al. 1999]. Essentially, this language is an extension of  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma})$  with assignments and a while loop. The use of similar languages will be illustrated in Section 5. We also refer to Gyssens et al. [1999]; and Van den Bussche [2000] for illustrations.

*Definition 2.18.* Let  $\sigma$  be a spatio-temporal database schema. Syntactically, a *program* in the language  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma}) + \text{while}$  is a finite (ordered) sequence of *statements* and *while-loops*. It is assumed there is a sufficient supply of new relation variables, each with an appropriate arity.

- (i) Each statement has the form

$$\overline{R} := \{(x_1, \dots, x_k) \mid \varphi(x_1, \dots, x_k)\};$$

Here,  $\overline{R}$  is a new relation variable with assigned arity  $k$  (the variables  $x_i$  range over  $\mathbf{R}$ ) and  $\varphi$  is a formula in  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma}')$ , where  $\overline{\sigma}'$  is the set of relation names containing the elements of  $\overline{\sigma}$  together with the relation variables introduced in previous statements of the program. Here, a previous statement is a statement appearing before the current statement in the sequence of statements which constitute the program.

- (ii) A while-loop has the form

**while  $\varphi$  do  $P$  end while**

where  $P$  is a program and  $\varphi$  is a sentence in  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma}')$ , where  $\overline{\sigma}'$  is again the set of relation names containing the elements of  $\overline{\sigma}$  together with the relation variables introduced in previous statements of the program.

- (iii) One of the relation names occurring in the program is designated as the output relation and is named  $\overline{R}_{\text{out}}$ .

Semantically, a program in the query language  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma}) + \text{while}$  expresses a spatio-temporal query as soon as  $\overline{R}_{\text{out}}$  is assigned a return value. The execution of an  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma}) + \text{while}$ -program applied to an input database is performed step-by-step. A statement is executed by first evaluating the  $\text{FO}(+, \times, <, 0, 1, \overline{\sigma})$ -formula on the right-hand side on the input database together with the new relations resulting from previous statements. Next, the result of the evaluation of the right-hand side is assigned to the relation variable

on the left-hand side. The effect of a while loop is to execute the body as long as the condition  $\varphi$  evaluates to true.

Note that these programs are not guaranteed to halt. For those input databases it does not, the query represented by the program is not defined on that particular input database.

### 3. SPATIO-TEMPORAL GENERICITY

As stated in the introduction, we are interested in spatio-temporal database queries that are invariant under the elements of a certain spatio-temporal transformation group (for function composition)

$$\mathcal{F} = \{f \mid f = (f_1, f_2, \dots, f_n, f_t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \times \mathbf{R}\}.$$

The idea is that the result of spatio-temporal queries should be largely independent of the particular coordinate system in which the data are presented. In this section, we formalize this idea by the notion of  $\mathcal{F}$ -genericity.

In this section, we look at different types of transformation groups and we impose two further conditions on these transformations. First, we look at purely temporal conditions. Second, we look at purely spatial or spatio-temporal conditions that reflect the nature of the queries one is interested in. We also look at transformation groups that are suited for applications in which physical notions such as velocity and acceleration are of importance.

#### 3.1 Definition of Spatio-Temporal Genericity

Let  $f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \times \mathbf{R}$  be a function, let  $R$  be a spatio-temporal relation name of arity  $k$  and let  $R^{\mathcal{F}}$  be a relation instance with underlying dimension  $n$ . In the following, we use the notation  $f(R^{\mathcal{F}})$  to abbreviate the set  $\{(f(\mathbf{a}_1, \tau_1), f(\mathbf{a}_2, \tau_2), \dots, f(\mathbf{a}_k, \tau_k)) \in (\mathbf{R}^n \times \mathbf{R})^k \mid (\mathbf{a}_1, \tau_1, \mathbf{a}_2, \tau_2, \dots, \mathbf{a}_k, \tau_k) \in R^{\mathcal{F}}\}$ .

*Definition 3.1.* Let  $Q$  be a spatio-temporal database query that takes databases of signature  $\sigma = \{R_1, \dots, R_m\}$  with underlying dimension  $n$  as input. Let  $\mathcal{F} = \{f \mid f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \times \mathbf{R}\}$  be a spatio-temporal transformation group. We say that  $Q$  is  $\mathcal{F}$ -generic if, for any  $f$  in  $\mathcal{F}$  and for each pair of spatio-temporal databases  $\mathcal{F}_1$  and  $\mathcal{F}_2$  over  $\sigma$ , the fact that  $\mathcal{F}_2 = (R_1^{\mathcal{F}_2}, \dots, R_m^{\mathcal{F}_2}) = (f(R_1^{\mathcal{F}_1}), \dots, f(R_m^{\mathcal{F}_1}))$  implies that  $f(Q(\mathcal{F}_1)) = Q(\mathcal{F}_2)$ .

This definition will be illustrated in Section 3.5.

It is clear that if a query is  $\mathcal{F}$ -generic, it is also  $\mathcal{F}'$ -generic for any subgroup  $\mathcal{F}'$  of  $\mathcal{F}$ .

#### 3.2 Temporal Restrictions on the Transformations

It is very natural to describe spatio-temporal events with the notions “before”, “after” and “co-temporal”. For instance, when two people arrive shortly after each other, we say “*Mary arrived before Jane*” rather than “*Mary arrived at 9:31 and Jane at 9:35*”. Another example is any kind of race. The winner is the one that finishes first. So, foremost the order of arrival of the participants matters. Exact-time moments are only important in very specific situations.

We start with the definition of a spatio-temporal event.

*Definition 3.2.* An event is a subset of  $\mathbf{R}^n \times \mathbf{R}$ . The projection of an event  $A$  on the time-axis is denoted by  $\pi_t(A)$  and called the *time-domain* of  $A$ .

Let  $A$  and  $B$  be events. In the terminology of Allen's interval calculus [Allen 1983; Allen and Ferguson 1994],  $A$  and  $B$  are called *co-temporal* if  $\pi_t(A) = \pi_t(B)$  (we denote this by  $A =_t B$ ). Allen says  $A$  is *before*  $B$  if  $t_A < t_B$  for all  $t_A \in \pi_t(A)$  and all  $t_B \in \pi_t(B)$  (we denote this by  $A <_t B$ ).

Remark that  $A \leq_t B := (A =_t B \text{ or } A <_t B)$  is a pre-order on events.

*Definition 3.3.* We say that a transformation  $f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \times \mathbf{R}$  preserves the order of events if for all events  $A$  and  $B$ ,  $A =_t B$  implies  $f(A) =_t f(B)$  and  $A <_t B$  implies  $f(A) <_t f(B)$ .

**PROPOSITION 3.4.** A transformation  $f = (f_1, f_2, \dots, f_n, f_t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \times \mathbf{R} : (\mathbf{x}, t) \mapsto (f_1(\mathbf{x}, t), \dots, f_n(\mathbf{x}, t), f_t(\mathbf{x}, t))$  preserves the order of events if and only if  $f_t$  is a strictly monotone increasing bijection of  $t$  alone.

**PROOF.** The if-direction is straightforward. To prove the other direction, let  $f = (f_1, f_2, \dots, f_n, f_t)$  be a transformation of  $\mathbf{R}^n \times \mathbf{R}$ . Consider any two events  $A = \{(a_1, a_2, \dots, a_n, \tau)\}$  and  $B = \{(a'_1, a'_2, \dots, a'_n, \tau)\}$ . Since  $A =_t B$ , then  $f_t(a_1, a_2, \dots, a_n, \tau) = f_t(a'_1, a'_2, \dots, a'_n, \tau)$ . This shows that  $f_t$  is a function of  $t$  alone.

Consider any two events  $A = \{(a_1, a_2, \dots, a_n, \tau_1)\}$  and  $B = \{(a_1, a_2, \dots, a_n, \tau_2)\}$  with  $\tau_1 < \tau_2$ . Since  $A <_t B$ , then  $f_t(\tau_1) < f_t(\tau_2)$ . This shows that  $f_t$  is a strictly monotone function of  $t$ .

The transformation groups that we consider are all groups with respect to the composition operator  $\circ$  of functions. Therefore, for every transformation  $f$  also its inverse exists, and hence  $f$  is a bijection. Given the fact that the component  $f_t$  is a function of  $t$  alone, it has to be a bijection too.  $\square$

We require that transformations preserve the order of events. We can therefore write the transformation groups of interest as a product of groups, that is,  $\mathcal{F} = (\mathcal{F}_{st}, \mathcal{F}_t)$ , where

$$(\mathcal{F}_{st}, \mathcal{F}_t) = \{(f_{st}, f_t) \mid f_{st} = (f_1, f_2, \dots, f_n) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \text{ and } f_t : \mathbf{R} \rightarrow \mathbf{R}\}.$$

The particular groups  $\mathcal{F}_t$  that we will consider in this article are:

- $\mathcal{A}_t = \{t \mapsto at + b \mid a, b \in \mathbf{R} \text{ and } a > 0\}$ , that is, the monotone affinities of the timeline;
- $\mathcal{T}_t = \{t \mapsto t + b \mid b \in \mathbf{R}\}$ , that is, the translations of the timeline; and
- $\text{Id}_t = \{\text{id}\}$ , that is, the identity of time.

Invariance with respect to this type of transformations of time is often encountered in physics [Desloge 1982].

### 3.3 Spatial and Spatio-Temporal Restrictions on Transformations

In the following, we consider transformations coming from practical situations where moving objects are monitored from a fixed position or situations where a

fixed object is observed from a moving position. The frame of reference is therefore changing in a time-dependent way. In real life, this continuous change of reference system arises in different kinds of situations. For example, when a moving person is watching an event, his/her description of that event will be related to his/her position and orientation at each time moment. When this person moves along a straight line at constant speed, the transformation that describes this continuous change of reference system would be a time-dependent affinity.

In this article, we only look at transformations that have an algebraic description. The general form of the transformation groups  $\mathcal{F}_{st}$  that we consider have elements of the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ t \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{11}(t) & \alpha_{12}(t) & \cdots & \alpha_{1n}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) & \cdots & \alpha_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1}(t) & \alpha_{n2}(t) & \cdots & \alpha_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \vdots \\ \beta_n(t) \end{pmatrix},$$

where the  $\alpha_{ij}$  and  $\beta_i$  are functions from  $\mathbf{R}$  to  $\mathbf{R}$ . Furthermore, we require that the transformation groups that we consider are “semi-algebraic” (we give a precise definition in Section 5.3).

We will consider the following groups  $\mathcal{F}_{st}$  of transformations:

- $\mathcal{A}_{st}$  is the group of transformations of the above form where the  $\alpha_{ij}(t)$  and  $\beta_i(t)$  are arbitrary functions of  $t$  such that the matrix of the  $\alpha_{ij}(t)$  has an inverse for each value of  $t$ , that is, these are the *time-dependent affinities*;
- $\mathcal{A}_{st}^f$  is the subgroup of  $\mathcal{A}_{st}$  consisting of transformations for which the functions  $\alpha_{ij}(t)$  and  $\beta_i(t)$  only take a finite number of values, that is, functions that are piecewise constant;
- $\mathcal{A}_{st}^c$  is the subgroup of  $\mathcal{A}_{st}^f$  consisting of transformations for which the functions  $\alpha_{ij}(t)$  and  $\beta_i(t)$  are constant;
- $\mathcal{S}_{st}, \mathcal{S}_{st}^f$  and  $\mathcal{S}_{st}^c$  are subgroups of  $\mathcal{A}_{st}, \mathcal{A}_{st}^f$  and  $\mathcal{A}_{st}^c$  respectively, where the matrix of the  $\alpha_{ij}(t)$  represents at each moment a similarity, that is, the composition of an orthogonal matrix and a scaling (given by a non-zero multiple of the unit matrix);
- $\mathcal{I}_{st}, \mathcal{I}_{st}^f, \mathcal{I}_{st}^c$  are the subgroups of the above groups where the matrix consisting of the  $\alpha_{ij}(t)$  is an orthogonal matrix, that is, these are groups of isometries.
- $\mathcal{T}_{st}, \mathcal{T}_{st}^f, \mathcal{T}_{st}^c$  are the subgroups of the above groups where the matrix consisting of the  $\alpha_{ij}(t)$  is the identity matrix, that is, these are groups of translations.

The choice of these transformation groups is to some extent arbitrary. These groups extend the transformation groups encountered in physics (see Section 3.4) in a natural way. Furthermore, they have the advantage that they can be defined in a semi-algebraic way. But it should be clear that other groups, like time-dependent homeomorphisms, could be considered.

### 3.4 Physical Transformation Groups

The following groups are of interest when notions such as velocity, acceleration and force are important in an application. These transformation groups can be found by solving the differential equations that express that these physical entities are preserved [Desloge 1982].

We define the velocity (respectively, acceleration) of a moving figure as the velocity (respectively, acceleration) of its center of mass. Changes in the shape of a moving figure during motion clearly affect the motion of the center of mass.

In case that we consider an arbitrary motion of a figure, the velocity and acceleration of the center of mass are preserved by the following two groups:

- $\mathcal{V}_{st}$  is the subgroup of  $\mathcal{A}_{st}^c$  where the  $\beta_i$  are constants. This group of transformations preserves the *velocity vector* of a moving figure.
- $\mathcal{AC}_{st}$  is the group  $\mathcal{A}_{st}^c$ . This group of transformations preserves the *acceleration vector* of a moving figure.

However, when we only consider figures which are in *rigid motion* or *isometric movement*, that is, a motion that preserves the shape of a figure, we require the transformations to preserve the rigidity as well. As a consequence we obtain the following two groups:

- $\mathcal{V}(\mathcal{R})_{st}$  is the subgroup of  $\mathcal{I}_{st}^c$  where the  $\beta_i$  are constants. This group of transformations preserves the *velocity vector* of a moving figure in rigid motion.
- $\mathcal{AC}(\mathcal{R})_{st}$  is the group  $\mathcal{I}_{st}^c$ . This group of transformations preserves the *acceleration vector* of a moving figure in rigid motion.

In physics it is customary to consider only translations for what concerns the time dimension, that is, the transformations in the group  $\mathcal{T}_t$ . The group  $(\mathcal{AC}(\mathcal{R})_{st}, \mathcal{T}_t)$  is also known as *the group of the Galilei transformations*. It is particularly useful because all laws of classical mechanics are invariant for this group of transformations of space-time [Desloge 1982].

### 3.5 Examples of Generic Queries

We end this section with a number of examples of queries that are generic for some of the genericity classes that we have introduced above.

Suppose in some city, an experiment is set up to evaluate the traffic situation. A number of probe cars (for simplicity, we assume two) is continuously driving around the city in a random way. The trajectories of the cars are stored in a spatio-temporal database, of underlying dimension 2, with schema  $\sigma = \{carA, carB\}$ , where the relations *carA* and *carB* both have arity 1. In these examples, we assume that time is measured in seconds and distance is measured in meters. We now give some example queries, and indicate for each the transformation groups it is generic for.

*Example 3.5.*  $Q_1$  : *Does the route followed by car A self-intersect more often than the route followed by car B does?*

This query is  $(\mathcal{V}_{st}, \mathcal{A}_t)$ -generic, but not  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic, for instance. It is not expressible in first-order logic. In Section 5, we will give a “program” expressing this query.



*Example 3.6.*  $Q_2$  : Give the places and time moments where it is true for car A that when it reaches them, it is standing still at that spot for at least 300 more seconds, (i.e., where and when did car A encounter a traffic jam?).

This query is  $(\mathcal{V}_{st}, \mathcal{T}_t)$ -generic. Indeed, the fact that a car has speed zero (when it is standing still) has to be preserved, which requires the group  $\mathcal{V}_{st}$ , and the length of time intervals has to be preserved, which requires  $\mathcal{T}_t$ . This query is expressed by the following FO(+, ×, <, 0, 1,  $\overline{carA}$ )-formula:

$$\varphi_2(x, y, t) := \overline{carA}(x, y, t) \wedge (\forall t_2)((t \leq t_2 \wedge t_2 \leq t + 300) \rightarrow \overline{carA}(x, y, t_2)).$$

*Example 3.7.*  $Q_3$  : Was there a collision between car A and car B?

This query is  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic and is expressed by the following FO(+, ×, <, 0, 1,  $\overline{carA}$ ,  $\overline{carB}$ )-formula:

$$\varphi_3 := (\exists x)(\exists y)(\exists t)(\overline{carA}(x, y, t) \wedge \overline{carB}(x, y, t)).$$

*Example 3.8.*  $Q_4$  : Did car A pass at 500 meters north of car B at time moment  $t = 5930$ ?

This query is  $(\mathcal{T}_{st}, \mathcal{I}_t)$ -generic and is expressed by the following FO(+, ×, <, 0, 1,  $\overline{carA}$ ,  $\overline{carB}$ )-formula:

$$\varphi_4 := (\exists x_1)(\exists y_1)(\exists y_2)(\overline{carA}(x_1, y_1, 5930) \wedge \overline{carB}(x_1, y_2, 5930) \wedge y_1 = y_2 + 500).$$

*Example 3.9.*  $Q_5$  : Did car A encounter any “empty roads”? (I.e., were there parts of its trajectory where it could drive at constant speed in a straight line for at least 6000 seconds?)

This query is  $(\mathcal{AC}_{st}, \mathcal{T}_t)$ -generic. The fact that a car drives at constant speed (i.e., has an acceleration of zero) has to be preserved. Note that, because the car’s movement is a polynomial function of time, driving at constant speed means driving in a straight line. Query  $Q_5$  can be expressed by the following FO(+, ×, <, 0, 1,  $\overline{carA}$ )-formula:

$$\begin{aligned} \varphi_5 := & (\exists t_1)(\exists t_2)(\exists x_1)(\exists y_1)(\exists x_2)(\exists y_2)(\overline{carA}(x_1, y_1, t_1) \wedge \overline{carA}(x_2, y_2, t_2) \wedge \\ & t_2 = t_1 + 6000 \wedge (\forall t_3)((t_1 \leq t_3 \wedge t_3 \leq t_2) \rightarrow (\exists x_3)(\exists y_3)(\overline{carA}(x_3, y_3, t_3) \wedge \\ & (t_2 - t_1)x_3 = (t_2 - t_3)x_1 + (t_3 - t_1)x_2 \wedge (t_2 - t_1)y_3 = (t_2 - t_3)y_1 + (t_3 - t_1)y_2))). \end{aligned}$$

This completes the examples section. We return to these examples later on, when we have defined point languages.

#### 4. SOUND AND COMPLETE LANGUAGES FOR THE GENERIC FIRST-ORDER SPATIO-TEMPORAL QUERIES

In this section, we study the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic queries that are expressible in FO(+, ×, <, 0, 1), for all groups  $(\mathcal{F}_{st}, \mathcal{F}_t)$  listed in Section 3.3 and Section 3.4.

To start with, we give a general undecidability result. We prove that it is undecidable whether a query is  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic, for any nontrivial group  $(\mathcal{F}_{st}, \mathcal{F}_t)$ .

The strategy to prove this result was introduced by Paredaens et al. [1994]. Let  $\mathbf{N}$  denote the set of the natural numbers.

**THEOREM 4.1.** *For all nontrivial groups  $(\mathcal{F}_{st}, \mathcal{F}_t)$  mentioned in the previous section,  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -genericity of spatio-temporal  $\text{FO}(+, \times, <, 0, 1, \sigma)$ -queries is undecidable, where  $\sigma$  is a non-empty schema.*

**PROOF.** Let  $\mathcal{F}$  be a group of transformations of  $\mathbf{R}^n \times \mathbf{R}$  that contains an element  $f_0$  that does not map  $(0, 0)$  to itself. We show that  $\mathcal{F}$ -genericity of spatio-temporal queries over a certain schema  $\sigma = \{R\}$ , where  $R$  is a one-dimensional unary spatio-temporal relation, of output type  $(1, 1)$  is undecidable. For other non-empty schemas the proof is similar. We will do this by reducing deciding the truth of sentences of the  $\forall^*$ -fragment of number theory to the genericity question. The  $\forall^*$ -fragment of number theory is known to be undecidable since Hilbert's 10th problem [Matijasevich 1993] can be formulated in it.

We encode a natural number  $n$  by the unary one-dimensional spatio-temporal relation

$$\text{enc}(n) := \{(0, 0), (1, 0), \dots, (n, 0)\}.$$

A  $k$ -dimensional vector of natural numbers  $(n_1, n_2, \dots, n_k)$  is encoded by the relation

$$\begin{aligned} \text{enc}(n_1, n_2, \dots, n_k) := & \text{enc}(n_1) \cup (\text{enc}(n_2) + (n_1 + 2, 0)) \cup \\ & \dots \cup (\text{enc}(n_k) + (n_1 + 2 + \dots + n_{k-1} + 2, 0)). \end{aligned}$$

For fixed  $k$ , the corresponding decoding is expressible in  $\text{FO}(+, \times, <, 0, 1)$ . We thus associate to the first-order sentence  $(\forall n_1) \dots (\forall n_k) \varphi(n_1, \dots, n_k)$  of number theory to the following spatio-temporal query  $Q_\varphi$  over the input schema  $\sigma = \{R\}$ :

```

if  $\bar{R}$  encodes a vector  $(n_1, \dots, n_k) \in \mathbf{N}^k$ 
then
  if  $\varphi(n_1, \dots, n_k)$ 
  then return  $\emptyset$ 
  else return  $\{(0, 0)\}$ 
end if
else return  $\emptyset$ 
end if

```

This query is expressible in  $\text{FO}(+, \times, <, 0, 1)$ .

**CLAIM.** *The query  $Q_\varphi$  is  $\mathcal{F}$ -generic if and only if the sentence  $(\forall n_1) \dots (\forall n_k) \varphi(n_1, \dots, n_k)$  holds in the natural numbers.*

**PROOF:** Now, we prove this claim. First, suppose that, for all  $(n_1, \dots, n_k) \in \mathbf{N}^k$ ,  $\varphi(n_1, \dots, n_k)$  holds. Let  $R$  be a one-dimensional unary spatio-temporal relation and let  $f$  be some transformation of  $\mathcal{F}$ . We have to prove that

$$f(Q_\varphi(R)) = Q_\varphi(f(R)).$$

The result of  $Q_\varphi(R)$  will always be  $\emptyset$ : either  $R$  does not encode a vector  $(n_1, \dots, n_k)$ , or it does and  $\varphi(n_1, \dots, n_k)$  holds. For the same reason,  $Q_\varphi(f(R))$  also equals  $\emptyset$ . The transformation  $f$  maps  $\emptyset$  to  $\emptyset$ , hence  $f(Q_\varphi(R)) = \emptyset$ , which concludes the first part of the proof.

Now assume that there exists an  $(n_1^0, \dots, n_k^0)$  such that  $\varphi(n_1^0, \dots, n_k^0)$  is not true. Let  $R$  be the database that decodes  $(n_1^0, \dots, n_k^0)$ . The result of  $Q_\varphi(R)$  will be the origin  $(0, 0)$  of  $\mathbf{R} \times \mathbf{R}$ . If we now apply  $f_0$  to this result, the output is a vector  $(y, z) \neq (0, 0)$ . On the other side, if we first apply  $f_0$  to  $R$ , there are three possibilities. Either  $f_0(R)$  encodes a vector  $(n_1^1, \dots, n_k^1)$  for which  $\varphi(n_1^1, \dots, n_k^1)$  is true, then the result of  $Q_\varphi(f_0(R))$  will be  $\emptyset$ . Or,  $f_0(R)$  encodes a vector  $(n_1^1, \dots, n_k^1)$  for which  $\varphi(n_1^1, \dots, n_k^1)$  is not true, and  $Q_\varphi(f_0(R))$  returns  $(0, 0)$ . In the last case,  $f_0(R)$  does not encode a vector of natural numbers, in which case the result of  $Q_\varphi(f_0(R))$  will be  $\emptyset$  again. In all cases, we have that  $Q_\varphi(f_0(R)) \neq f_0(Q_\varphi(R))$ . Therefore, the query  $Q_\varphi$  is not  $\mathcal{F}$ -generic.

We can conclude that  $Q_\varphi$  is  $\mathcal{F}$ -generic if and only if the sentence  $(\forall n_1) \dots (\forall n_k) \varphi(n_1, \dots, n_k)$  holds in the natural numbers.

Therefore, if  $\mathcal{F}$ -genericity would be decidable, also the truth of sentences in the  $\forall^*$ -fragment of number theory would be decidable. This concludes the proof.  $\square$

Next, we show that  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic FO(+,  $\times$ ,  $<$ , 0, 1)-queries are recursive enumerable, however. We do this by syntactically specifying sound and complete languages that capture the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic FO-queries. These languages are point-based logics defined as follows:

*Definition 4.2.* Let  $\sigma = \{R_1, R_2, \dots, R_m\}$  be a spatio-temporal database schema and let  $\Pi$  be a set of predicates. The first-order logic over  $\sigma$  and  $\Pi$ , denoted by FO( $\Pi, R_1, R_2, \dots, R_m$ ) or FO( $\Pi, \sigma$ ) for short, can be used as a spatio-temporal query language when variables are interpreted to range over points in  $\mathbf{R}^n \times \mathbf{R}$ , (we denote variables by  $u, v, w, \dots$ ). The atomic formulas in FO( $\Pi, \sigma$ ) are equality constraints on point variables, the predicates of  $\Pi$  applied to point variables, and the relation names  $R_1, R_2, \dots, R_m$  from  $\sigma$  applied to point variables.

*Example 4.3.* Let  $p_1$  and  $p_2$  be points in  $\mathbf{R}^n \times \mathbf{R}$ . Let **Before**( $p_1, p_2$ ) be a predicate which evaluates to true if the time coordinate  $\tau_1$  of  $p_1$  is smaller than or equal to the time coordinate  $\tau_2$  of  $p_2$ . Let  $\Pi = \{\mathbf{Before}\}$  and consider the following formula **Cotemp** in FO( $\Pi$ ):

$$\mathbf{Cotemp}(u, v) := \mathbf{Before}(u, v) \wedge \mathbf{Before}(v, u).$$

This formula expresses that two points  $p_1$  and  $p_2$  are co-temporal, or equivalently, that  $\tau_1$  equals  $\tau_2$ .

A FO( $\Pi, \sigma$ )-formula  $\varphi(v_1, v_2, \dots, v_k)$  defines for each spatio-temporal database  $\mathcal{F}$  over  $\sigma$  a subset  $\varphi(\mathcal{F})$  of  $(\mathbf{R}^n \times \mathbf{R})^k$  defined as

$$\{(p_1, \dots, p_k) \in (\mathbf{R}^n \times \mathbf{R})^k \mid (\mathbf{R}^n \times \mathbf{R}, \Pi^{\mathbf{R}^n \times \mathbf{R}}, R_1^{\mathcal{F}}, \dots, R_m^{\mathcal{F}}) \models \varphi[p_1, \dots, p_k]\},$$

where  $\varphi[p_1, \dots, p_k]$  is obtained from the formula  $\varphi(v_1, \dots, v_k)$  by instantiating the variables  $v_i$  by the constant points  $p_i$ ,  $1 \leq i \leq k$ .

From Definition 2.14, it is clear what it means that a FO( $\Pi, \sigma$ )-formula *expresses a semi-algebraic databases query*.

*Definition 4.4.* A query language is said to be *sound* for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic FO(+,  $\times$ ,  $<$ , 0, 1)-queries on spatio-temporal databases, if that language only

expresses  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic  $\text{FO}(+, \times, <, 0, 1)$ -queries on spatio-temporal databases.

A query language is said to be *complete* for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic  $\text{FO}(+, \times, <, 0, 1)$ -queries on spatio-temporal databases, if all  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic  $\text{FO}(+, \times, <, 0, 1)$ -queries on spatio-temporal databases can be expressed in that language.

In the following, we will omit the dependence on the input schema when this is clear from the context, and use the notation  $\text{FO}(\Pi)$  for first-order point languages over the predicate set  $\Pi$ .

In the remainder of this section, we give sound and complete point languages for (i) time-independent transformations (Theorem 4.10, Section 4.1), (ii) physics-related transformations (Theorem 4.12, Section 4.2), and (iii) time-dependent transformations (Theorem 4.20, Section 4.3).

#### 4.1 Genericity for Time-Independent Transformations

In this section, we give a general result concerning  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic queries where  $\mathcal{F}_{st}$  is a subgroup of  $\mathcal{A}_{st}^c$ , the group of time-independent affinities of  $\mathbf{R}^n \times \mathbf{R}$ . First, we introduce the point predicates that we will use for the different point languages.

To express the temporal order of events, we use the point predicate **Before** defined in Example 4.3. We will also often use the derived predicate **Cotemp**, which expresses the co-temporality of events. The definition of **Cotemp** is given in Example 4.3 as well.

There are three more other purely temporal predicates: **UnitTime**,  $\mathbf{0}_t$  and  $\mathbf{1}_t$ . The predicate **UnitTime** $(p_1, p_2)$  expresses that the points  $p_1, p_2 \in \mathbf{R}^n \times \mathbf{R}$  have time-coordinates  $\tau_1$  and  $\tau_2$  such that  $|\tau_1 - \tau_2| = 1$ . The unary predicates  $\mathbf{0}_t$  and  $\mathbf{1}_t$  are such that  $\mathbf{0}_t(p)$  and  $\mathbf{1}_t(p)$  respectively express that the time coordinate of the point  $p$  equals to zero and to one.

The following predicates address spatio-temporal relations between points. The point-predicate **Between** $^{n+1}$  is defined such that **Between** $^{n+1}(p_1, p_2, p_3)$  expresses that the points  $p_1, p_2, p_3$  in  $\mathbf{R}^n \times \mathbf{R}$  are collinear (in the space  $\mathbf{R}^n \times \mathbf{R}$ ) and that  $p_2$  is between  $p_1$  and  $p_3$ . The predicates  $\leq_i(p_1, p_2)$  ( $1 \leq i \leq n$ ) express that the  $i$ th spatial coordinate of  $p_1$  is less or equal than the  $i$ th spatial coordinate of  $p_2$ . The expression **EqDist** $(p_1, p_2, p_3, p_4)$  is true if the distance between the two co-temporal points  $p_1$  and  $p_2$  equals the distance between the two co-temporal points  $p_3$  and  $p_4$ . The binary predicate **UnitDist** applied to two points  $p_1$  and  $p_2$  expresses that they are co-temporal and that the (spatial) distance between  $p_1$  and  $p_2$  equals one. Finally, **Pos** $^{n+1}(p_0, p_1, p_2, \dots, p_{n+1})$  expresses that the  $(n + 2)$ -tuple  $(p_0, p_1, p_2, \dots, p_{n+1})$  of points in  $\mathbf{R}^n \times \mathbf{R}$  forms a positively oriented  $(n + 1)$ -dimensional coordinate system with  $p_0$  as origin.

The proof of the following property consists of a straightforward encoding of the point predicates in  $\text{FO}(+, \times, <, 0, 1)$  similar to the encodings presented in Gyssens et al. [1999]. We leave the proof as an exercise for the reader.

**PROPERTY 4.5.** *The point predicates **Before**, **Between** $^{n+1}$ , **UnitTime**,  $\mathbf{0}_t$ ,  $\mathbf{1}_t$ ,  $\leq_i$  ( $1 \leq i \leq n$ ), **EqDist**, **UnitDist** and **Pos** $^{n+1}$  are all expressible in  $\text{FO}(+, \times, <, 0, 1)$ .*

Table I. An Overview of the Different Sets of Point Predicates for a Number of Spatio-Temporal Time-Independent Genericity Notions. In the Three Last Cases  $\mathcal{F}_t \in \{\mathcal{A}_t, \mathcal{T}_t, \text{Id}_t\}$

| $(\mathcal{F}_{st}, \mathcal{F}_t)$   | Sets of Point Predicates $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  |
|---------------------------------------|--|
| $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ | $\{\mathbf{Between}^{n+1}, \mathbf{Before}\}$  |
| $(\mathcal{A}_{st}^c, \mathcal{T}_t)$ | $\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}\}$   |
| $(\mathcal{A}_{st}^c, \text{Id}_t)$   | $\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}, \mathbf{0}_t, \mathbf{1}_t\}$                                       |
| $(\mathcal{S}_{st}^c, \mathcal{F}_t)$ | $\Pi(\mathcal{A}_{st}^c, \mathcal{F}_t) \cup \{\mathbf{EqDist}\}$  |
| $(\mathcal{I}_{st}^c, \mathcal{F}_t)$ | $\Pi(\mathcal{A}_{st}^c, \mathcal{F}_t) \cup \{\mathbf{EqDist}, \mathbf{UnitDist}\}$   |
| $(\mathcal{T}_{st}^c, \mathcal{F}_t)$ | $\Pi(\mathcal{A}_{st}^c, \mathcal{F}_t) \cup \{\mathbf{EqDist}, \mathbf{UnitDist}, \leq_i (1 \leq i \leq n), \mathbf{Pos}^{n+1}\}$ |

PROPERTY 4.6. *Let  $(\mathcal{F}_{st}, \mathcal{F}_t)$  be a group and let  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  be a set of point predicates as in Table I. The point predicates in  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  are invariant under elements of  $(\mathcal{F}_{st}, \mathcal{F}_t)$ .*

PROOF. This is a straightforward consequence of the definitions of the point predicates and the properties of the different transformation groups. We refer to Section A.1 in the appendix for the complete proof.  $\square$

Remark 4.7. From now, all results are valid for underlying dimension  $n \geq 2$ .

We now identify sound and complete point languages for the time-independent transformations in Table I. Soundness and completeness are established as follows: First, we provide a “meta-theorem” (Theorem 4.8) that gives sufficient conditions for any point language containing **Between** <sup>$n+1$</sup>  to be sound and complete. Next, we specialize this theorem for point languages containing **Between** <sup>$n+1$</sup>  and **Before** and for time-independent transformations (Theorem 4.9). Finally, we show that the sufficient conditions are satisfied for the point languages and groups in Table I, and as a result obtain soundness and completeness (Theorem 4.10).

The following theorem follows directly from the proof of Theorem 5.5 [Gyssens et al. 1999]. We recall that the correspondence between a spatio-temporal database schema  $\sigma$  and its semi-algebraic counterpart  $\bar{\sigma}$  is defined in Remark 2.6.

THEOREM 4.8. *Let  $\sigma$  be a spatio-temporal database schema. Let  $\mathcal{F}$  be a subgroup of the affinities of  $\mathbf{R}^n \times \mathbf{R}$ . Let  $\Pi$  be a set of point-predicates that contains **Between** <sup>$n+1$</sup> . If the predicates in  $\Pi$  are  $\text{FO}(+, \times, <, 0, 1)$ -expressible and invariant under the transformations of  $\mathcal{F}$  and if the fact “ $(v_0, v_1, \dots, v_{n+1})$  is the image of the standard coordinate system of  $\mathbf{R}^n \times \mathbf{R}$  under some element  $f$  of  $\mathcal{F}$ ” is expressible in  $\text{FO}(\Pi)$ , then  $\text{FO}(\Pi, \sigma)$  is sound and complete for the  $\mathcal{F}$ -generic spatio-temporal database queries that are expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

We now prove the following theorem.

THEOREM 4.9. *Let  $\sigma$  be a spatio-temporal database schema. Let  $\mathcal{F}_{st}$  be a subgroup of  $\mathcal{A}_{st}^c$  and  $\mathcal{F}_t$  a subgroup of  $\mathcal{A}_t$ . Let  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  be a set of point-predicates that contains **Between** <sup>$n+1$</sup>  and **Before**. If the predicates in  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  are  $\text{FO}$ -expressible and invariant under the transformations of  $(\mathcal{F}_{st}, \mathcal{F}_t)$  and if*

the fact “ $(v_0, v_1, \dots, v_{n+1})$  is the image of the standard coordinate system under some element  $f$  of  $(\mathcal{F}_{st}, \mathcal{F}_t)$ ” is expressible in  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t))$ , then the logic  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma)$  is sound and complete for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic spatio-temporal database queries that are expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .

**PROOF.** First, we show that the language  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}\})$  is sound and complete for the  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ -generic  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -expressible spatio-temporal database queries using Theorem 4.8. Indeed, it is clear that the group  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$  is a subgroup of the affinities of  $\mathbf{R}^n \times \mathbf{R}$ . Furthermore, the expression  $\mathbf{Before}(u, v)$ , is expressible in  $\text{FO}(+, \times, <, 0, 1)$  (see Property 4.5). Also, the predicates  $\mathbf{Between}^{n+1}$  and  $\mathbf{Before}$  are both invariant under elements of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$  (see Property 4.6).

To conclude this part of the proof, we need to show that there is an expression in  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}\})$  that, for  $n + 2$  arbitrary points  $p_0, p_1, \dots, p_{n+1}$  in  $\mathbf{R}^n \times \mathbf{R}$ , states that  $(p_0, p_1, \dots, p_{n+1})$  is the image of the standard coordinate system under some element  $f$  of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ . It is known (e.g. Gyssens et al. [1999] and Schwabhäuser et al. [1983]) that there exists an expression in the language  $\text{FO}(\{\mathbf{Between}^{n+1}\})$  that, for  $n + 2$  points  $p_0, p_1, \dots, p_{n+1}$  of  $\mathbf{R}^n \times \mathbf{R}$ , expresses that  $(p_0, p_1, \dots, p_{n+1})$  is the image of the standard  $(n + 1)$ -dimensional coordinate system under some affinity of  $\mathbf{R}^n \times \mathbf{R}$ . We refer to this expression as

$$\mathbf{CoSys}_{\mathcal{A}}(v_0, v_1, \dots, v_{n+1}).$$

Obviously, this formula also belongs to  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}\})$ . The expression for the image of the standard coordinate system under some element of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$  is as follows:

$$\begin{aligned} \mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{A}_t)}(v_0, v_1, \dots, v_{n+1}) &:= \mathbf{CoSys}_{\mathcal{A}}(v_0, v_1, \dots, v_{n+1}) \wedge \\ &\bigwedge_{i=1}^n \mathbf{Cotemp}(v_0, v_i) \wedge \neg \mathbf{Before}(v_{n+1}, v_0). \end{aligned}$$

It is easy to verify that any coordinate system that is an image of the standard coordinate system under an element of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$  satisfies this expression. Also, the reverse is true. For clarity, we show this only for  $n = 2$  (the general case is analogous).

Any coordinate system  $(p_0, p_1, p_2, p_3)$  satisfying the expression  $\mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{A}_t)}(v_0, v_1, v_2, v_3)$  is of the form  $p_0 = (a_{0,1}, a_{0,2}, \tau_0)$ ,  $p_1 = (a_{1,1}, a_{1,2}, \tau_0)$ ,  $p_2 = (a_{2,1}, a_{2,2}, \tau_0)$ ,  $p_3 = (a_{3,1}, a_{3,2}, \tau_3)$ , where  $\tau_0 < \tau_3$  and the determinant

$$\begin{vmatrix} a_{1,1} - a_{0,1} & a_{1,2} - a_{0,2} & 0 \\ a_{2,1} - a_{0,1} & a_{2,2} - a_{0,2} & 0 \\ a_{3,1} - a_{0,1} & a_{3,2} - a_{0,2} & \tau_3 - \tau_0 \end{vmatrix} \neq 0. \quad (*)$$

Now, we have to show that there exists an element  $f$  of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$  such that the image of the standard coordinate system under  $f$  equals  $(p_0, p_1, p_2, p_3)$ . As  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$  is a subgroup of the affinities,  $f$  is representable by a matrix. It is



straightforward to derive that  $f = (f_{st}, f_t)$ , where

$$f_{st}(x, y, t) = \begin{pmatrix} a_{1,1} - a_{0,1} & a_{2,1} - a_{0,1} \\ a_{1,2} - a_{0,2} & a_{2,2} - a_{0,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (a_{3,1} - a_{0,1})t + a_{0,1} \\ (a_{3,2} - a_{0,2})t + a_{0,2} \end{pmatrix}, \text{ and}$$

$$f_t(t) = (\tau_3 - \tau_0)t + \tau_0.$$

It is clear that  $(\tau_3 - \tau_0) > 0$  and that, because of the inequality (\*), the value of the determinant  $\begin{vmatrix} a_{1,1} - a_{0,1} & a_{2,1} - a_{0,1} \\ a_{1,2} - a_{0,2} & a_{2,2} - a_{0,2} \end{vmatrix}$  differs from zero, hence  $f$  is an element of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ .

So far, we proved that the language  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}\}, \sigma)$  is sound and complete for the  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ -generic queries expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ . The fact that any other language  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma)$ , where  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  contains  $\mathbf{Between}^{n+1}$  and  $\mathbf{Before}$ , is sound and complete for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -queries for each subgroup  $(\mathcal{F}_{st}, \mathcal{F}_t)$  of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ , under the conditions stated in Theorem 4.9, follows from Theorem 4.8 together with the first part of this proof.  $\square$

**THEOREM 4.10.** *Let  $\sigma$  be a spatio-temporal database schema. Let  $(\mathcal{F}_{st}, \mathcal{F}_t)$  be a group and let  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  be as in Table I. The point language  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma)$  is sound and complete for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic queries expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

**PROOF.** We can apply Theorem 4.9 for all groups in Table I, because they are all subgroups of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ . From Properties 4.5 and 4.6, it follows that all predicates are expressible in  $\text{FO}(+, \times, <, 0, 1)$  and that they are invariant under transformations of the appropriate groups. The only thing left to prove is that, for all groups  $(\mathcal{F}_{st}, \mathcal{F}_t)$  from Table I, and for  $n + 2$  points  $v_0, v_1, \dots, v_{n+1}$  in  $\mathbf{R}^n \times \mathbf{R}$ , the fact “ $(v_0, v_1, \dots, v_{n+1})$  is the image of the standard coordinate system under some element  $f$  of  $(\mathcal{F}_{st}, \mathcal{F}_t)$ ” is expressible in  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t))$ . For each group  $(\mathcal{F}_{st}, \mathcal{F}_t)$  from Table I, we now give a formula that expresses this fact. The correctness of these formulas is easy to verify.

—For the group  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ , we already gave a formula, denoted by  $\mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{A}_t)}$ , expressing the desired fact in the proof of Theorem 4.9.

—For the group  $(\mathcal{A}_{st}^c, \mathcal{T}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{T}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{A}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge \mathbf{UnitTime}(v_0, v_{n+1}).$$

—For the group  $(\mathcal{A}_{st}^c, \mathcal{Id}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{Id}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{T}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge \mathbf{0}_t(v_0) \wedge \mathbf{1}_t(v_{n+1}).$$

Let  $\mathcal{F}_t$  be an element of  $\{\mathcal{A}_t, \mathcal{T}_t, \mathcal{Id}_t\}$ .

—For the groups  $(\mathcal{S}_{st}^c, \mathcal{F}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{S}_{st}^c, \mathcal{F}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{A}_{st}^c, \mathcal{F}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge \bigwedge_{i=1}^n \bigwedge_{j=1}^n \mathbf{EqDist}(v_0, v_i, v_0, v_j).$$

Table II. An Overview of the Different Point-Predicate Sets for the Physics-Related Transformation Groups

| $(\mathcal{F}_{st}, \mathcal{T}_t)$               | Set of point predicates $\Pi(\mathcal{F}_{st}, \mathcal{T}_t)$   |
|---|--|
| $(\mathcal{V}_{st}, \mathcal{T}_t)$               | $\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}, =_{\text{space}}\}$                                     |
| $(\mathcal{V}(\mathcal{R})_{st}, \mathcal{T}_t)$  | $\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}, =_{\text{space}}, \mathbf{EqDist}, \mathbf{UnitDist}\}$ |
| $(\mathcal{AC}_{st}, \mathcal{T}_t)$              | $\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}\}$   |
| $(\mathcal{AC}(\mathcal{R})_{st}, \mathcal{T}_t)$ | $\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}, \mathbf{EqDist}, \mathbf{UnitDist}\}$                   |

—For the groups  $(\mathcal{I}_{st}^c, \mathcal{F}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{I}_{st}^c, \mathcal{F}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{S}_{st}^c, \mathcal{F}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge \bigwedge_{i=1}^n \mathbf{UnitDist}(v_0, v_i).$$

—For the groups  $(\mathcal{T}_{st}^c, \mathcal{F}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{T}_{st}^c, \mathcal{F}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{I}_{st}^c, \mathcal{F}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge \mathbf{Pos}^{n+1}(v_0, v_1, \dots, v_{n+1}) \wedge \bigwedge_{j=1}^n \bigwedge_{i=1}^n \leq_i(v_0, v_j).$$

This concludes the proof.  $\square$

## 4.2 Genericity for Physics-Related Transformations

Here, we focus on the physics-related transformation groups  $(\mathcal{V}_{st}, \mathcal{T}_t)$ ,  $(\mathcal{V}(\mathcal{R})_{st}, \mathcal{T}_t)$ ,  $(\mathcal{AC}_{st}, \mathcal{T}_t)$  and  $(\mathcal{AC}(\mathcal{R})_{st}, \mathcal{T}_t)$ . To formulate our results, we need to define one more point-predicate, namely  $=_{\text{space}}$ . If  $p_1 = (a_{1,1}, \dots, a_{1,n}, \tau_1)$  and  $p_2 = (a_{2,1}, \dots, a_{2,n}, \tau_2)$  are elements of  $\mathbf{R}^n \times \mathbf{R}$ , then  $=_{\text{space}}(p_1, p_2)$  if and only if  $a_{1,i} = a_{2,i}$  for all  $1 \leq i \leq n$ .

*Remark 4.11.* The expression

$$=_{\text{space}}(v_1, v_2) := \bigwedge_{i=1}^n (\leq_i(v_1, v_2) \wedge \leq_i(v_2, v_1))$$

is expressible in  $\text{FO}(+, \times, <, 0, 1)$ .

**THEOREM 4.12.** *Let  $\sigma$  be a spatio-temporal database schema. Let the groups  $(\mathcal{F}_{st}, \mathcal{T}_t)$  and the predicate sets  $\Pi(\mathcal{F}_{st}, \mathcal{T}_t)$  be as in Table II. The point language  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{T}_t), \sigma)$  is sound and complete for the  $(\mathcal{F}_{st}, \mathcal{T}_t)$ -generic spatio-temporal queries that are expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

**PROOF.** The transformation groups  $(\mathcal{F}_{st}, \mathcal{T}_t)$  of Table II are all subgroups of the group  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ . Furthermore, the predicates of  $\Pi(\mathcal{F}_{st}, \mathcal{T}_t)$  are expressible in  $\text{FO}(+, \times, <, 0, 1)$  (see Property 4.5 and Remark 4.11). Straightforward geometrical and physical arguments show that all predicates are invariant under the appropriate transformation groups. We can now apply Theorem 4.9. We only have to verify that it is possible to express in the languages  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{T}_t))$  that a coordinate system is the image of the standard  $(n+1)$ -dimensional coordinate

system under an element of  $(\mathcal{F}_{st}, \mathcal{T}_t)$ . We now give, for each group  $(\mathcal{F}_{st}, \mathcal{T}_t)$  from Table II, the expression for the fact that  $(v_0, v_1, \dots, v_{n+1})$  is the image of the standard coordinate system under some element  $f$  of  $(\mathcal{F}_{st}, \mathcal{T}_t)$ .

The correctness of these expressions is easy to verify.

—For the group  $(\mathcal{V}_{st}, \mathcal{T}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{V}_{st}, \mathcal{T}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{A}_{st}, \mathcal{T}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge =_{\mathbf{space}}(v_0, v_{n+1}),$$

because elements of this group map the origin  $(0, \dots, 0, 0)$  and the unit vector in the time-direction  $(0, \dots, 0, 1)$  of the standard coordinate system of  $\mathbf{R}^n \times \mathbf{R}$  onto points which have equal spatial coordinates.

—For the group  $(\mathcal{V}(\mathcal{R})_{st}, \mathcal{T}_t)$ , we have

$$\mathbf{CoSys}_{(\mathcal{V}(\mathcal{R})_{st}, \mathcal{T}_t)}(v_0, v_1, \dots, v_{n+1}) := \mathbf{CoSys}_{(\mathcal{I}_{st}, \mathcal{T}_t)}(v_0, v_1, \dots, v_{n+1}) \wedge =_{\mathbf{space}}(v_0, v_{n+1}).$$

—For the groups  $(\mathcal{AC}_{st}, \mathcal{T}_t)$  and  $(\mathcal{AC}(\mathcal{R})_{st}, \mathcal{T}_t)$  we can simply use the expressions for  $(\mathcal{AC}_{st}, \mathcal{T}_t)$  and  $(\mathcal{I}_{st}, \mathcal{T}_t)$ , respectively, given in the proof of Theorem 4.10.  $\square$

Next, we illustrate the languages summarized in Table I and Table II on the appropriate examples of Section 3.5.

*Example 4.13.* We give the  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}, =_{\mathbf{space}}\})$ -query  $Q'_2$  equivalent to the  $(\mathcal{V}_{st}, \mathcal{T}_t)$ -generic query of Example 3.6: *Give the places and time moments where car A is standing still at that spot for at least 300 more seconds.*

Remember that we assumed before that time is measured in seconds and distance is measured in meters. We first remark that the fact that one point is a constant number of seconds before another, can be expressed using **UnitTime** and **Before**. We illustrate this for an easy example where one point is 3 seconds after another:

$$3\text{sec}(u, v) := (\exists w_1)(\exists w_2)(\mathbf{Before}(u, w_1) \wedge \mathbf{Before}(w_1, w_2) \wedge \mathbf{Before}(w_2, v) \wedge \mathbf{UnitTime}(u, w_1) \wedge \mathbf{UnitTime}(w_1, w_2) \wedge \mathbf{UnitTime}(w_2, v)).$$

Now we give the expression for  $Q'_2$ :

$$\text{carA}(u) \wedge (\exists v)(300\text{sec}(u, v) \wedge (\forall w)((\mathbf{Before}(u, w) \wedge \mathbf{Before}(w, v) \wedge \text{carA}(w)) \rightarrow =_{\mathbf{space}}(u, w))).$$

*Example 4.14.* We give the  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}, \mathbf{UnitTime}\})$ -query  $Q'_5$  equivalent to the  $(\mathcal{AC}_{st}, \mathcal{T}_t)$ -generic query of Example 3.9: *Did car A encounter any empty roads? That is, were there parts of its trajectory where it could drive at constant speed for at least 6000 seconds.*

$$(\exists u)(\exists v)(\text{carA}(u) \wedge \text{carA}(v) \wedge 6000\text{sec}(u, v) \wedge (\forall w)((\text{carA}(w) \wedge \mathbf{Before}(u, w) \wedge \mathbf{Before}(w, v)) \rightarrow (\mathbf{Between}^{n+1}(u, v, w)))).$$

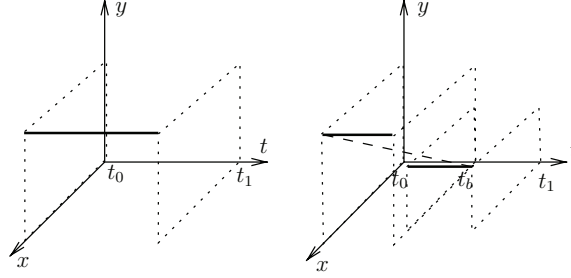


Fig. 2. The elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  do not preserve betweenness of points.

### 4.3 Genericity for Time-Dependent Transformations

Here, we focus on notions of genericity determined by time-dependent transformations. We will focus first on time-dependent transformations which are piecewise constant, that is,  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  and its subgroups introduced in Section 3.3. Later we show how to deal with the arbitrary, that is, not piecewise-constant, time-dependent transformation groups, that is,  $(\mathcal{A}_{st}, \mathcal{A}_t)$  and its subgroups introduced in Section 3.3.

It is important to note that for  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  and its subgroups, we cannot apply Theorem 4.9. Indeed, it heavily relies on the fact that, using the predicate **Between** <sup>$n+1$</sup> , it can be expressed that  $n + 2$  points form an affine coordinate system for the space  $\mathbf{R}^n \times \mathbf{R}$ , and also that some points represent the coordinates of another point, relative to such an affine coordinate system (the latter is a straightforward consequence of the former). When using the transformation group  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  or one of its subgroups, the predicate **Between** <sup>$n+1$</sup>  is too strong. Indeed, transformations of the group  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  do not preserve “betweenness” in  $(n + 1)$ -dimensional space of points with different time coordinates. Therefore, the notion of collinearity in  $(n + 1)$ -dimensional space can no longer be used. Figure 2 illustrates this with a line (left) and the image of the line under some transformation  $\alpha = (\alpha_{st}, \alpha_t)$  in  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  for which  $\alpha_t$  is the identity function and  $\alpha_{st}$  equals the identity in the time interval  $[t_0, t_b[$  and is a constant translation of space for the interval  $[t_b, t_1]$ . In the left part of Figure 2, it is true that all points different from the endpoints at time moments  $t_0$  and  $t_1$  lie between the endpoints. For the right part of Figure 2, this is not true (the dashed line connecting the end points indicates all points between them.)

However, as we want our language to be able to express all first-order  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic queries, somehow there needs to be a link between an  $(n + 1)$ -dimensional point and its coordinates. It will become more clear later that, although we cannot express projection along the time axis, this link can be expressed using the predicates **Between** <sup>$n$</sup> , **Before** and a new predicate, **EqCR**<sup>st</sup>. The predicate **Before** has already been introduced in Section 4.1. The expression **Between** <sup>$n$</sup>  $(p, q, r)$  states, for three points  $p, q, r \in \mathbf{R}^n \times \mathbf{R}$ , that they are co-temporal, collinear in the space  $\mathbf{R}^n$  and that  $q$  is between  $p$  and  $r$ . We also introduce a new 6-ary predicate, **EqCR**<sup>st</sup>. For six points  $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbf{R}^n \times \mathbf{R}$ , **EqCR**<sup>st</sup> $(p_1, p_2, p_3, q_1, q_2, q_3)$  expresses that the cross ratio of the three co-temporal and collinear points  $p_1, p_2$  and  $p_3$  equals the cross ratio of the time

coordinates  $\tau_{q_1}$ ,  $\tau_{q_2}$  and  $\tau_{q_3}$  of the points  $q_1$ ,  $q_2$  and  $q_3$ . The cross ratio of three collinear points  $p, q, r$  is  $\frac{|pq|}{|pr|}$ , where  $|pq|$  denotes the length of the line segment between  $p$  and  $q$ . It is well known that the cross ratio is invariant under affine transformations.

For example, in  $\mathbf{R}^2 \times \mathbf{R}$ ,

$$\mathbf{EqCR}^{\text{st}}((0, 0, 0), (1, 1, 0), (2, 2, 0), (0, 0, 0), (0, 0, 1), (0, 0, 2))$$

holds, since the former three points have a cross ratio of  $\frac{\sqrt{2}}{2\sqrt{2}}$  and the latter three points have a cross ratio of  $\frac{1}{2}$ .

For ease of use, we will often use the predicates  $\mathbf{EqCR}^{\text{s}}$  for the cross-ratio of spatial coordinates, and  $\mathbf{EqCR}^{\text{t}}$  for the cross-ratio of temporal coordinates. Both predicates can be expressed using  $\mathbf{EqCR}^{\text{st}}$ :

$$\begin{aligned} \mathbf{EqCR}^{\text{s}}(u_1, u_2, u_3, v_1, v_2, v_3) &:= (\exists w_1)(\exists w_2)(\exists w_3) \\ &(\mathbf{EqCR}^{\text{st}}(u_1, u_2, u_3, w_1, w_2, w_3) \wedge \mathbf{EqCR}^{\text{st}}(v_1, v_2, v_3, w_1, w_2, w_3)), \end{aligned}$$

and

$$\begin{aligned} \mathbf{EqCR}^{\text{t}}(u_1, u_2, u_3, v_1, v_2, v_3) &:= (\exists w_1)(\exists w_2)(\exists w_3) \\ &(\mathbf{EqCR}^{\text{st}}(w_1, w_2, w_3, u_1, u_2, u_3) \wedge \mathbf{EqCR}^{\text{st}}(w_1, w_2, w_3, v_1, v_2, v_3)). \end{aligned}$$

For the remainder of this section, we will assume that  $\Pi$  denotes the set

$$\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\},$$

unless stated otherwise.

Next, we present the main theorem of this section.

**THEOREM 4.15.** *Let  $\sigma$  be a spatio-temporal database schema. The language  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\}, \sigma)$  is sound and complete for the  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic spatio-temporal queries that are expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

**PROOF.** We prove this theorem by three lemmas. First, the soundness is addressed in Lemma 4.16. Next, we prove completeness in two steps: Lemma 4.17 shows that every  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -formula can be converted into a  $\text{FO}(\Pi, \sigma)$ -formula, parameterized by a set of coordinate systems and Lemma 4.18 shows then that every  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic spatio-temporal query that is expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$  can be converted into an equivalent query expressible in the language  $\text{FO}(\Pi, \sigma)$ . Taken together, these three lemma's prove the theorem.  $\square$

It remains to prove the three lemma's. First, we establish the soundness of  $\text{FO}(\Pi, \sigma)$  for piecewise constant time-dependent transformation groups.

**LEMMA 4.16.** *Let  $\sigma$  be a spatio-temporal database schema and let  $n$  be the underlying dimension. The language  $\text{FO}(\Pi, \sigma)$  is sound for the  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic spatio-temporal queries expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

**PROOF.** Soundness is proved in two steps. First, we show that every  $\text{FO}(\Pi, \sigma)$ -formula is equivalently expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$  and afterwards that every  $\text{FO}(\Pi, \sigma)$ -formula is invariant under elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ .

Both are proved by induction on the structure of  $\text{FO}(\Pi, \sigma)$ -formulas. We refer to Section A.2 of the appendix for the complete proof.  $\square$

We now show that every  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -formula can be converted into a  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\}, \sigma)$ -formula, which is parameterized by a finite set of coordinate systems.

A coordinate system in a  $n$ -dimensional hyperplane of  $\mathbf{R}^n \times \mathbf{R}$ , orthogonal to the time axis will be referred to as a *spatial coordinate system* and a coordinate system on the time-axis will be referred to as a *temporal coordinate system*.

If  $p, q$  and  $r$  are collinear points in  $\mathbf{R}^n \times \mathbf{R}$ , then we denote by  $\frac{\overrightarrow{pq}}{\overrightarrow{pr}}$  the real number  $\alpha$  such that  $\overrightarrow{pq} = \alpha \overrightarrow{pr}$ .

**LEMMA 4.17.** *Let  $\sigma$  be a spatio-temporal database schema and let the underlying dimension be  $n$ . For every  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -formula*

$$\bar{\psi}(x_1, x_2, \dots, x_m, t_1, \dots, t_l),$$

*there exists a  $\text{FO}(\Pi, \sigma)$ -formula*

$$\psi(u_{t_0}, u_{t_E}, u_{0,0}, u_{0,1}, \dots, u_{0,n}, \dots, u_{l,0}, u_{l,1}, \dots, u_{l,n}, v_1, v_2, \dots, v_k),$$

*where  $l$  is the number of variables occurring in the formula that refer to a time dimension and where  $k$  is the total number of free variables of  $\bar{\psi}$ , that is,  $k = m + l$ .*

*Furthermore, for each spatio-temporal database  $\mathcal{F}$  over  $\sigma$ , for each set of spatial coordinate systems  $(p_{i,0}, p_{i,1}, \dots, p_{i,n})$ ,  $i = 0, \dots, l$  of the spatial component of  $\mathbf{R}^n \times \mathbf{R}$ , for each temporal coordinate system  $(p_{t_0}, p_{t_E})$  of the temporal component of  $\mathbf{R}^n \times \mathbf{R}$ , and for all points  $q_1, q_2, \dots, q_k$  on the line  $p_{0,0}p_{0,1}$ :*

$$(\mathbf{R}^n \times \mathbf{R}, \Pi^{\mathbf{R}^n \times \mathbf{R}}, \mathcal{F}) \models \psi[p_{t_0}, p_{t_E}, p_{0,0}, p_{0,1}, \dots, p_{0,n}, \dots, p_{l,0}, p_{l,1}, \dots, p_{l,n}, q_1, q_2, \dots, q_k]$$

*if and only if*

$$(\mathbf{R}, +, \times, <, 0, 1, \alpha(\bar{\mathcal{F}})) \models \bar{\psi} \left[ \frac{\overrightarrow{p_{0,0}q_1}}{\overrightarrow{p_{0,0}p_{0,1}}}, \frac{\overrightarrow{p_{0,0}q_2}}{\overrightarrow{p_{0,0}p_{0,1}}}, \dots, \frac{\overrightarrow{p_{0,0}q_k}}{\overrightarrow{p_{0,0}p_{0,1}}} \right],$$

*where  $\alpha = (\alpha_{st}, \alpha_t)$  is an element of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  such that  $(p_{0,0}, \dots, p_{0,n})$  is mapped by  $\alpha_{st}$  onto the standard spatial coordinate system in the hyperplane  $\mathbf{R}^n \times \{(0, \dots, 0, 0)\}$  of  $\mathbf{R}^n \times \mathbf{R}$ , and each spatial coordinate system  $(p_{i,0}, p_{i,1}, \dots, p_{i,n})$  ( $i = 1, \dots, l$ ) is mapped on the standard coordinate system in the hyperplane at time  $\mathbf{R}^n \times \{\alpha(\tau_{p_{i,0}})\}$  where the temporal part  $\alpha_t$  of  $\alpha$  is the unique time-affinity that maps  $\tau_{p_0}$  to 0 and  $\tau_{p_E}$  to 1.*

**PROOF.** We present the structure of the proof here and refer to Section A.3 in the appendix for the complete proof.

Let  $\bar{\psi}$  be a  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -formula. We assume that  $\bar{\psi}$  is in prenex normal form. The proof constructs a translation of  $\bar{\psi}$  into a formula  $\psi$  of  $\text{FO}(\Pi, \sigma)$  (by induction on its structure). In this translation, first the quantifier-free part of  $\bar{\psi}$  is translated.



To start with, a 2-dimensional “computation plane” is chosen in some  $n$ -dimensional coordinate system that is used to simulate real variables, constants and all the polynomial equations, polynomial equalities and inequalities.

Next, relation predicates  $\bar{R}$  in  $\bar{\psi}$  are translated into  $R$ . This is the most intricate part of the translation. With each point variable  $v$  in  $R$  an  $n$ -dimensional coordinate systems is associated, which is defined using  $n + 1$  point variables and is co-temporal with  $v$ . In this coordinate system,  $n$  other point variables represent the coordinates of  $v$  with respect to this coordinate system. Since all variables are already represented by point variables in the computation plane, we need to link the set of point variables in the computation plane with the point variables representing the coordinates of  $v$  in its associated coordinate system. This can be achieved by means of the predicate **EqCR<sup>s</sup>**. As a result, if  $R$  contains  $m$  point variables,  $m + 1$  coordinate systems are needed. This explains the occurrence of the parameters in the translation in the statement of the Lemma.

The composition of formulas requires some special attention to ensure that coordinate systems introduced in different formulas are in agreement with each other. Finally, quantifiers are added in the obvious way.  $\square$

**LEMMA 4.18.** *Let  $\sigma$  be a spatio-temporal database schema. For every  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic spatio-temporal query expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ , there exists an equivalent  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\}, \sigma)$ -query.*

**PROOF.** Given a  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic spatio-temporal query of output type  $(n, k)$ , expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ ,

$$\bar{\psi}(x_{1,1}, \dots, x_{1,n}, x_{1,t}, \dots, x_{k,1}, \dots, x_{k,n}, x_{k,t}).$$

The conversion procedure, given in Lemma 4.17, returns a formula

$$\psi(u_{t_0}, u_{t_E}, u_{0,0}, \dots, u_{0,n}, u_{1,0}, \dots, u_{1,n}, \dots, u_{k,0}, \dots, u_{k,n}, \\ v_{1,1}, \dots, v_{1,n}, v_{1,t}, \dots, v_{k,1}, \dots, v_{k,n}, v_{k,t}),$$

parameterized by one temporal and  $k$  spatial coordinate systems and which is, up to a transformation of the group  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ , that depends on the coordinate systems, equivalent to the original formula  $\bar{\psi}$ . Since it has additional free variables, the query  $\psi$  clearly has the wrong output type. A  $\text{FO}(\Pi, \sigma)$ -query equivalent to  $\bar{\psi}$  should be a formula

$$\psi_{\text{final}}(v_1, v_2, \dots, v_k)$$

having  $k$  free variables only. We obtain the desired formula by introducing  $k$  new point variables  $v_i$ , and for each  $1 \leq i \leq k$ ,  $n$  new point variables  $v'_{i,1}, \dots, v'_{i,n}$  such that  $v'_{i,j}$  is collinear with  $u_{i,0}$  and  $u_{i,1}$  and

$$\mathbf{Coordinates}^n(u_{i,0}, \dots, u_{i,n}, v'_{i,1}, \dots, v'_{i,n}, v_i). \quad (1)$$

Moreover, we require that

$$\mathbf{EqCR}^{\text{st}}(u_{0,0}, u_{0,1}, v_{i,t}, u_{t_0}, u_{t_E}, v_i) \quad (2)$$

and

$$\bigwedge_{j=1}^n \mathbf{EqCR}^s(u_{0,0}, u_{0,1}, v_{i,j}, u_{i,0}, u_{i,1}, v'_{i,j}). \quad (3)$$

The final formula  $\psi_{\text{final}}$  is now obtained by existentially quantifying all point variables, except for  $v_1, \dots, v_k$  in the conjunction of  $\psi$  with the expressions (1), (2) and (3).

Now consider the (partial) output of  $\psi_{\text{final}}$  when we choose a specific coordinate system for each set of variables  $u_{i,0}, \dots, u_{i,n}$ . By similar reasoning as in Lemma 4.17, we obtain that this partial output equals

$$\alpha'^{-1}(\overline{\psi}(\alpha(\overline{\mathcal{F}})))$$

where  $\alpha' = (\alpha'_{st}, \alpha'_t)$  and  $\alpha = (\alpha_{st}, \alpha_t)$  both are transformations as specified in the statement of Lemma 4.17. This means that they both satisfy the same set of constraints, that is,  $\alpha'_t = \alpha_t$  and for certain time moments  $\tau$ ,  $\alpha'_{st}(\tau) = \alpha_{st}(\tau)$ . In between those time moments  $\alpha'_{st}$  and  $\alpha_{st}$  can differ. However, it follows from Lemma 4.17 that for any two transformations  $\alpha$  and  $\alpha'$  satisfying the constraints as described in the statement of Lemma 4.17,  $(\mathbf{R}, +, \times, <, 0, 1, \alpha(\overline{\mathcal{F}})) \models \overline{\psi}[\frac{p_{0,0}q_1}{p_{0,0}p_{0,1}}, \frac{p_{0,0}q_2}{p_{0,0}p_{0,1}}, \dots, \frac{p_{0,0}q_k}{p_{0,0}p_{0,1}}]$  if and only if  $(\mathbf{R}^n \times \mathbf{R}, \Pi^{\mathbf{R}^n \times \mathbf{R}}, \mathcal{F}) \models \psi[p_{t_0}, p_{t_E}, p_{0,0}, p_{0,1}, \dots, p_{0,n}, \dots, p_{l,0}, p_{l,1}, \dots, p_{l,n}, q_1, q_2, \dots, q_k]$  if and only if  $(\mathbf{R}, +, \times, <, 0, 1, \alpha'(\overline{\mathcal{F}})) \models \overline{\psi}[\frac{p_{0,0}q_1}{p_{0,0}p_{0,1}}, \frac{p_{0,0}q_2}{p_{0,0}p_{0,1}}, \dots, \frac{p_{0,0}q_k}{p_{0,0}p_{0,1}}]$ . In other words, we have that  $\overline{\psi}(\alpha(\overline{\mathcal{F}})) = \overline{\psi}(\alpha'(\overline{\mathcal{F}}))$ . Hence, we can conclude without loss of generality that the partial output of  $\psi_{\text{final}}$  when we fill in a specific coordinate system for each set of variables  $u_{i,0}, \dots, u_{i,n}$  equals  $\alpha^{-1}(\overline{\psi}(\alpha(\overline{\mathcal{F}})))$  where  $\alpha$  is a transformation as specified in the statement of Lemma 4.17.

If we now consider all possible coordinate systems for each set of variables  $u_{i,0}, \dots, u_{i,n}$ , then we have

$$\psi_{\text{final}}(\mathcal{F}) = \bigcup_c \bigcup_{\alpha_c} (\alpha_c^{-1}(\overline{\psi}(\alpha_c(\overline{\mathcal{F}})))),$$

where  $c$  ranges over all possible coordinate system assignments and  $\alpha_c$  ranges over all transformations satisfying the constraints following from this choice of coordinate systems.

The union  $\bigcup_c \bigcup_{\alpha_c} (\alpha_c^{-1}(\overline{\psi}(\alpha_c(\overline{\mathcal{F}}))))$  is in fact the union over all elements  $\alpha$  of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  of  $\alpha^{-1}(\overline{\psi}(\alpha(\overline{\mathcal{F}})))$ . So,

$$\psi_{\text{final}}(\mathcal{F}) = \bigcup_{\alpha} (\alpha^{-1}(\overline{\psi}(\alpha(\overline{\mathcal{F}}))),$$

where  $\alpha$  ranges over all elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ .

Since  $\psi_{\text{final}}$  is a  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic query and the group  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  is semi-algebraic (we give a precise definition in Section 5.3), we have that for every  $\alpha$ ,

$$\alpha^{-1}(\overline{\psi}(\alpha(\overline{\mathcal{F}}))) = \overline{\psi}(\overline{\mathcal{F}}).$$

So, finally,

$$\psi_{\text{final}}(\mathcal{F}) = \overline{\psi}(\overline{\mathcal{F}}).$$

This concludes the proof.  $\square$

Table III. An Overview of the Different Sets of Point Predicate for Some Transformation Groups. We have  $\mathcal{F}_t \in \{\mathcal{A}_t, \mathcal{T}_t, Id_t\}$ 

| $(\mathcal{F}_{st}, \mathcal{F}_t)$       | Sets of point predicates $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  |
|---|--|
| $(\mathcal{A}_{st}^{(f)}, \mathcal{A}_t)$ | <b>{Between<sup>n</sup>, Before, EqCR<sup>st</sup>}</b>  |
| $(\mathcal{A}_{st}^{(f)}, \mathcal{T}_t)$ | <b>{Between<sup>n</sup>, Before, EqCR<sup>st</sup>, UnitTime}</b>  |
| $(\mathcal{A}_{st}^{(f)}, Id_t)$          | <b>{Between<sup>n</sup>, Before, EqCR<sup>st</sup>, UnitTime, 0<sub>t</sub>, 1<sub>t</sub>}</b>  |
| $(\mathcal{S}_{st}^{(f)}, \mathcal{F}_t)$ | $\Pi(\mathcal{A}_{st}^{(f)}, \mathcal{F}_t) \cup \{\mathbf{EqDist}^{\text{cotemp}}\}$  |
| $(\mathcal{I}_{st}^{(f)}, \mathcal{F}_t)$ | $\Pi(\mathcal{A}_{st}^{(f)}, \mathcal{F}_t) \cup \{\mathbf{EqDist}^{\text{cotemp}}, \mathbf{UnitDist}\}$   |
| $(\mathcal{T}_{st}^{(f)}, \mathcal{F}_t)$ | $\Pi(\mathcal{A}_{st}^{(f)}, \mathcal{F}_t) \cup \{\mathbf{EqDist}^{\text{cotemp}}, \mathbf{UnitDist}, \leq_i (1 \leq i \leq n), \mathbf{Pos}^n\}$ |

As mentioned at the beginning of this section, Theorem 4.15 only relates to piecewise constant transformations groups. However, a careful analysis of the proof of Lemma 4.17 shows that we only consider a finite number of moments in time (i.e., there are only a finite number of time variables in any  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -formula  $\varphi$ ). This implies that the transformation  $\mathcal{A}_{st}^f$  and  $\mathcal{A}_{st}$  yield the same results. So, we can use the proof given above for the group  $(\mathcal{A}_{st}, \mathcal{A}_t)$ . Indeed, in between the moments of time that are considered, it is not important which transformation function is used.

As a result, Theorem 4.15 applies to the transformation groups  $(\mathcal{A}_{st}, \mathcal{A}_t)$  as well:

**COROLLARY 4.19.** *Let  $\sigma$  be a spatio-temporal database schema. The language  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\}, \sigma)$  is sound and complete for the  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic spatio-temporal queries that are expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

Corollary 4.19 has the following corollary. Before we can state this corollary, we need to define two extra point predicates, namely  $\mathbf{EqDist}^{\text{cotemp}}$  and  $\mathbf{Pos}^n$ .

First,  $\mathbf{EqDist}^{\text{cotemp}}(p, q, r, s)$  is true for four co-temporal points  $p, q, r$  and  $s$  if and only if the (Euclidean) distance between  $p$  and  $q$  equals the distance between  $r$  and  $s$ . Second, the expression  $\mathbf{Pos}^n(p_0, p_1, \dots, p_n)$  is true for  $n + 1$  co-temporal points  $p_0, p_1, \dots, p_n$  if and only if  $(p_0, p_1, \dots, p_n)$  forms a positively oriented coordinate system.

**COROLLARY 4.20.** *Let  $\sigma$  be a spatio-temporal database schema. Let  $(\mathcal{F}_{st}, \mathcal{F}_t)$  and  $\text{FO}(\mathcal{F}_{st}, \mathcal{F}_t)$  be taken from Table III. The language  $\mathcal{P}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma)$  is sound and complete for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic spatio-temporal queries that are expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ .*

**PROOF.** It follows directly from the proof of Theorem 4.15 that, for each subgroup  $(\mathcal{F}_{st}^{(f)}, \mathcal{F}_t)$  of  $(\mathcal{A}_{st}^{(f)}, \mathcal{A}_t)$ , the language  $\text{FO}(\Pi, \sigma)$  is sound and complete for the  $(\mathcal{F}_{st}^{(f)}, \mathcal{F}_t)$ -generic queries expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$  if and only if the following three conditions are satisfied:

- (i) the set  $\Pi$  contains the elements **Between<sup>n</sup>, Before** and **EqCR<sup>st</sup>**;
- (ii) all elements of  $\Pi$  are  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ -expressible and invariant under the transformations of  $(\mathcal{F}_{st}^{(f)}, \mathcal{F}_t)$ ;
- (iii) the facts “ $(v_0, v_1, \dots, v_n)$  is the image of the standard coordinate system in the hyperplane co-temporal with  $v_{i_0}$  under an element of  $(\mathcal{F}_{st}^{(f)}, \mathcal{F}_t)$ ” and

“( $v_{t_O}, v_{t_E}$ ) is the image of the standard temporal coordinate system under an element of  $(\mathcal{F}_{st}^{(f)}, \mathcal{F}_t)$ ”, where  $v_0, v_1, \dots, v_n, v_{t_O}$  and  $v_{t_E}$  are points in  $(n + 1)$ -dimensional real space, are expressible in  $\text{FO}(\Pi)$ .

All groups listed in Table III are subgroups of  $(\mathcal{A}_{st}^{(f)}, \mathcal{A}_t)$  and satisfy the first condition. It is also straightforward to verify that they satisfy the second condition.

For the third condition, we list for every group mentioned in Table III the expressions for the spatial and temporal coordinate system. The proof that these expressions are correct is straightforward.

—For the group  $(\mathcal{A}_{st}^{(f)}, \mathcal{A}_t)$ , the expressions for  $\mathbf{TCoSys}_{\mathcal{A}}(u_1, u_2)$  and  $\mathbf{CoSys}_{\mathcal{A}}^n(u_0, u_1, \dots, u_n)$  were given in Lemma 4.17.

—For the group  $(\mathcal{A}_{st}^{(f)}, \mathcal{I}_t)$ , the expression for the spatial coordinate system does not change, but

$$\mathbf{TCoSys}_{\mathcal{T}}(u_1, u_2) := \mathbf{TCoSys}_{\mathcal{A}}(u_1, u_2) \wedge \mathbf{UnitTime}(u_1, u_2).$$

—For the group  $(\mathcal{A}_{st}^{(f)}, Id_t)$ , the expression for the spatial coordinate system does again not change, but

$$\mathbf{TCoSys}_{Id}(u_1, u_2) := \mathbf{TCoSys}_{\mathcal{T}}(u_1, u_2) \wedge \mathbf{0}_t(u_1) \wedge \mathbf{1}_t(u_2).$$

For the following groups, we only list the expression for the spatial coordinate system. The temporal coordinate system depends on the groups  $\mathcal{F}_t$  and is completely analogous to the previous cases.

—For the group  $(\mathcal{S}_{st}^{(f)}, \mathcal{F}_t)$ , we have

$$\mathbf{CoSys}_{\mathcal{S}}(u_0, u_1, \dots, u_n) :=$$

$$\mathbf{CoSys}_{\mathcal{A}}^n(u_0, u_1, \dots, u_n) \wedge \bigwedge_{i=1}^n \bigwedge_{j=1}^n \mathbf{EqDist}^{\text{cotemp}}(u_0, u_i, u_0, u_j).$$

—For the group  $(\mathcal{I}_{st}^{(f)}, \mathcal{F}_t)$ , we have

$$\mathbf{CoSys}_{\mathcal{I}}(u_0, u_1, \dots, u_n) := \mathbf{CoSys}_{\mathcal{S}}(u_0, u_1, \dots, u_n) \wedge \bigwedge_{i=1}^n \mathbf{UnitDist}(u_0, u_i).$$

—For the group  $(\mathcal{T}_{st}^{(f)}, \mathcal{F}_t)$ , we have

$$\mathbf{CoSys}_{\mathcal{T}}(u_0, u_1, \dots, u_n) :=$$

$$\mathbf{CoSys}_{\mathcal{I}}(u_0, u_1, \dots, u_n) \wedge \bigwedge_{i=1}^n \bigwedge_{j=1}^n \leq_i(u_0, u_j) \wedge \mathbf{Pos}^n(u_0, u_1, \dots, u_n).$$

This concludes the proof.  $\square$

Next, we illustrate the languages summarized in Table III with the appropriate examples of Section 3.5.

*Example 4.21.* We give the  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\})$ -expression  $\varphi'_3$  equivalent to the  $(\mathcal{I}_{st}, \mathcal{A}_t)$ -generic query of Example 3.7: *Was there a collision between car A and car B?*:

$$\varphi'_3 := (\exists u)(\text{carA}(u) \wedge \text{carB}(u)).$$

Remark that this query can be expressed without the use of the point predicates from  $\Pi$ .

*Example 4.22.* We give the  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}, \mathbf{EqDist}^{\text{cotemp}}, \mathbf{UnitDist}, \leq_i (1 \leq i \leq n), \mathbf{Pos}^n, \mathbf{UnitTime}, \mathbf{0}_t, \mathbf{1}_t\})$ -expression  $\varphi'_4$  equivalent to the  $(\mathcal{I}_{st}, \text{Id}_t)$ -generic query of Example 3.8: *Did car A pass at 500 meters north of car B at time moment  $t = 5930$ ?*

The fact that a point has time coordinate 5930 can be expressed using  $\mathbf{UnitTime}$ ,  $\mathbf{0}_t$ , and  $\mathbf{1}_t$ . We illustrate this with a predicate expressing the fact that a point has time coordinate 3:

$$\text{eq}3t(u) := (\exists v_1)(\exists v_2)(\mathbf{1}_t(v_1) \wedge \mathbf{Before}(v_1, v_2) \wedge \mathbf{UnitTime}(v_1, v_2) \wedge \mathbf{Before}(v_2, u) \wedge \mathbf{UnitTime}(v_2, u)).$$

The fact that the distance between two points is 500 can be expressed using  $\mathbf{UnitDist}$  in a way comparable to the construction of the predicate  $\text{3sec}$  of Example 4.13.

Now we give the expression  $\varphi'_4$ :

$$(\exists u)(\exists v)(\exists w)(\text{carA}(u) \wedge \text{carB}(v) \wedge \text{eq}5930t(u) \wedge \text{eq}5930t(v) \wedge (\leq_1(u, w) \wedge \leq_1(w, u)) \wedge (\leq_2(v, w) \wedge \leq_2(w, v)) \wedge \text{500meters}(u, w)).$$

## 5. SOUND AND COMPLETE LANGUAGES FOR THE COMPUTABLE GENERIC SPATIO-TEMPORAL QUERIES

In this section, we show that the languages  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma)$  of the previous section, when extended with assignment statements and a while loop, yield languages that are computationally sound and complete for the computable queries that are  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic. To start with, we explain in more detail how point-based logics are extended with assignment statements and a while loop. Afterwards, this section is organized in the same way as Section 4. We first discuss sound and complete languages for the queries generic for time-independent transformation groups. Then we focus on genericity for groups related to physical notions. Finally, we address sound and complete languages for the queries that are generic for the time-dependent transformations.

We start with extending the point-based logics described in Definition 4.2 with while loops.

*Definition 5.1.* Let  $\Pi$  be a finite set of point predicates, and let  $\sigma$  be a database schema. Syntactically, a *program* in the language  $\text{FO}(\Pi, \sigma) + \text{while}$  is a finite sequence of *statements* and *while-loops*. It is assumed there is a sufficient supply of new relation variables, each with an appropriate arity.

- (i) Each statement has the form

$$R := \{(u_1, \dots, u_k) \mid \varphi(u_1, \dots, u_k)\};.$$

Here,  $R$  is a new relation variable with assigned arity  $k$  (the variables  $u_i$  range over  $\mathbf{R}^n \times \mathbf{R}$ ) and  $\varphi$  is a formula in  $\text{FO}(\Pi, \sigma')$ , where  $\sigma'$  is the set of relation names containing the elements of  $\sigma$  together with the relation variables introduced in previous statements of the program.

(ii) A while-loop has the form

**while**  $\varphi$  **do**  $P$  **end while**

where  $P$  is a program and  $\varphi$  is a sentence in  $\text{FO}(\Pi, \sigma')$ , where  $\sigma'$  is again the set of relation names containing the elements of  $\sigma$  together with the relation variables introduced in previous statements of the program.

(iii) One of the relation names occurring in the program is designated as the output relation and is named  $R_{\text{out}}$ .

Semantically, a program in the query language  $\text{FO}(\Pi, \sigma) + \text{while}$  expresses a spatio-temporal query as soon as  $R_{\text{out}}$  is assigned a return value. The execution of a  $\text{FO}(\Pi, \sigma) + \text{while}$ -program applied to an input database is performed step-by-step. A statement is executed by first evaluating the  $\text{FO}(\Pi, \sigma)$ -formula on the right-hand side on the input database together with the newly created relations resulting from previous statements. Next, the result of the evaluation of the right-hand side is assigned to the relation variable on the left-hand side. The effect of a while loop is to execute the body as long as the condition  $\varphi$  evaluates to true.

Note that an  $\text{FO}(\Pi, \sigma) + \text{while}$ -program is not guaranteed to halt. For those input databases it does not, the query represented by the program is not defined on that particular input database.

Consider the following example which will be used later on to express the query from Example 3.5.

*Example 5.2.* Suppose that we have a spatio-temporal database with schema  $\sigma = \{R, S\}$ , where the underlying dimension is two and both  $R$  and  $S$  have arity one. We assume that all points in  $R$  and  $S$  have disjoint time coordinates. This means that we can sort all points according to their time coordinates. We also assume that  $R$  and  $S$  both contain a finite number of points.

The query  $Q$  we want to answer is the following: *Does  $R$  contain more points than  $S$ ?* It is well known that we cannot express this query in first-order logic [Grumbach and Su 1997]. The  $\text{FO}(\Pi, \sigma) + \text{while}$ -program expressing  $Q$  is:

```

 $R_{\text{Not}} := \{\};$ 
 $S_{\text{Not}} := \{\};$ 
 $R_{\text{Smallest}} := \{(u) | R(u) \wedge \neg R_{\text{Not}}(u) \wedge$ 
   $(\forall v)((R(v) \wedge \neg R_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v)))\};$ 
 $S_{\text{Smallest}} := \{(u) | S(u) \wedge \neg S_{\text{Not}}(u) \wedge$ 
   $(\forall v)((S(v) \wedge \neg S_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v)))\};$ 
while  $(\exists u)(R_{\text{Smallest}}(u)) \wedge (\exists v)(S_{\text{Smallest}}(v))$  do
   $R_{\text{Not}} := \{(u) | R_{\text{Not}}(u) \vee R_{\text{Smallest}}(u)\};$ 
   $S_{\text{Not}} := \{(u) | S_{\text{Not}}(u) \vee S_{\text{Smallest}}(u)\};$ 
   $R_{\text{Smallest}} := \{(u) | R(u) \wedge \neg R_{\text{Not}}(u) \wedge$ 
     $(\forall v)((R(v) \wedge \neg R_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v)))\};$ 
   $S_{\text{Smallest}} := \{(u) | S(u) \wedge \neg S_{\text{Not}}(u) \wedge$ 
     $(\forall v)((S(v) \wedge \neg S_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v)))\};$ 
end while
 $R_{\text{out}} := \{()\} | (\exists u)(R_{\text{Smallest}}(u));$ 

```



Intuitively, this program repeatedly takes the earliest point from both  $R$  and  $S$  until they do not both contain unvisited points anymore. When the while loop terminates and  $R$  still contains unvisited points, true is returned.

### 5.1 Genericity for Time-Independent Transformations

In this section, we prove a general result concerning computable  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic queries where  $(\mathcal{F}_{st}, \mathcal{F}_t)$  is a time-independent affinity of  $\mathbf{R}^n \times \mathbf{R}$ , that is, a group from Table I. The following theorem follows directly from the proof of Theorem 6.1 [Gyssens et al. 1999].

**THEOREM 5.3.** *Let  $\sigma$  be a spatio-temporal database schema. Let  $\mathcal{F}$  be a subgroup of the affinities of  $\mathbf{R}^n \times \mathbf{R}$ , let  $\Pi$  be a set of point predicates and let  $\text{FO}(\Pi, \sigma)$  be a point language that is sound and complete for the  $\mathcal{F}$ -generic queries expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ . Then, the language  $\text{FO}(\Pi) + \text{while}$  is sound and complete for the  $\mathcal{F}$ -generic computable queries.*

From this, we can derive the following result:

**COROLLARY 5.4.** *Let  $\sigma$  be a spatio-temporal database schema. Let  $(\mathcal{F}_{st}, \mathcal{F}_t)$  be a group and let  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  be as in Table I. The point language  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma) + \text{while}$  is sound and complete for the computable  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic queries over  $\bar{\sigma}$ .*

**PROOF.** The correctness follows from Theorem 4.10 and Theorem 5.3.  $\square$

### 5.2 Genericity for Physics-Related Transformations

Here, we focus again on the transformation groups  $(\mathcal{V}_{st}, \mathcal{I}_t)$ ,  $(\mathcal{V}(\mathcal{R})_{st}, \mathcal{I}_t)$ ,  $(\mathcal{AC}_{st}, \mathcal{I}_t)$  and  $(\mathcal{AC}(\mathcal{R})_{st}, \mathcal{I}_t)$ . As they are all subgroups of the affinities of  $\mathbf{R}^n \times \mathbf{R}$ , we can apply Theorem 5.3 again.

**COROLLARY 5.5.** *Let  $\sigma$  be a spatio-temporal database schema. Let  $(\mathcal{F}_{st}, \mathcal{I}_t)$  be a group from Table II and let  $\Pi(\mathcal{F}_{st}, \mathcal{I}_t)$  be as in Table II. The point language  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{I}_t), \sigma) + \text{while}$  is sound and complete for the computable spatio-temporal queries over  $\bar{\sigma}$  that are  $(\mathcal{F}_{st}, \mathcal{I}_t)$ -generic.*

**PROOF.** The correctness follows from Theorem 4.12 and Theorem 5.3.  $\square$

**Example 5.6.** We now give the  $\text{FO}(\{\mathbf{Between}^{n+1}, \mathbf{Before}, =_{\text{space}}\}) + \text{while}$ -program expressing query  $Q_1$  of Example 3.5: *Does the route followed by car A self-intersect more often than the route followed by car B does?*

If a car is standing still at a certain position, this will result in an infinite number of points in  $\mathbf{R}^n \times \mathbf{R}$  with the same spatial coordinates. However, one would not consider this situation to be an infinite number of self-intersections. Therefore, when such a situation happens, we only consider the last moment of the interval during which the car is at that specific location.

Intuitively, the program first computes the relations containing all self-intersections of the trajectories of both cars, and then determines whether the route of car A self-intersects the most. The program of Example 5.2 can

be used to perform this last task. We slightly adapt it such that it expresses query  $Q_1$ :

$$\begin{aligned}
A_{\cap} &:= \{ (u) | \text{car}A(u) \wedge (\exists v)(\text{car}A(v) \wedge \mathbf{Before}(u, v) \wedge =_{\text{space}}(u, v) \wedge \\
&\quad (\forall w)((\text{car}A(w) \wedge \mathbf{Before}(u, w) \wedge \mathbf{Before}(w, v) \wedge u \neq w \wedge v \neq w) \\
&\quad \rightarrow \neg(=_{\text{space}}(w, v))))); \\
B_{\cap} &:= \{ (u) | \text{car}B(u) \wedge (\exists v)(\text{car}B(v) \wedge \mathbf{Before}(u, v) \wedge =_{\text{space}}(u, v) \wedge \\
&\quad (\forall w)((\text{car}B(w) \wedge \mathbf{Before}(u, w) \wedge \mathbf{Before}(w, v) \wedge u \neq w \wedge v \neq w) \\
&\quad \rightarrow \neg(=_{\text{space}}(w, v))))); \\
A_{\text{Not}} &:= \{ \}; \\
B_{\text{Not}} &:= \{ \}; \\
A_{\text{Smallest}} &:= \{ (u) | A_{\cap}(u) \wedge \neg A_{\text{Not}}(u) \wedge (\forall v)( \\
&\quad (A_{\cap}(v) \wedge \neg A_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v))) \}; \\
B_{\text{Smallest}} &:= \{ (u) | B_{\cap}(u) \wedge \neg B_{\text{Not}}(u) \wedge (\forall v)( \\
&\quad (B_{\cap}(v) \wedge \neg B_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v))) \}; \\
\mathbf{while} & (\exists u)(A_{\text{Smallest}}(u)) \wedge (\exists v)(B_{\text{Smallest}}(v)) \mathbf{do} \\
&\quad A_{\text{Not}} := \{ (u) | A_{\text{Not}}(u) \vee A_{\text{Smallest}}(u) \}; \\
&\quad B_{\text{Not}} := \{ (u) | B_{\text{Not}}(u) \vee B_{\text{Smallest}}(u) \}; \\
&\quad A_{\text{Smallest}} := \{ (u) | A_{\cap}(u) \wedge \neg A_{\text{Not}}(u) \wedge (\forall v)( \\
&\quad (A_{\cap}(v) \wedge \neg A_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v))) \}; \\
&\quad B_{\text{Smallest}} := \{ (u) | B_{\cap}(u) \wedge \neg B_{\text{Not}}(u) \wedge (\forall v)( \\
&\quad (B_{\cap}(v) \wedge \neg B_{\text{Not}}(v)) \rightarrow (\mathbf{Before}(u, v))) \}; \\
\mathbf{end while} & \\
R_{\text{out}} &:= \{ () | (\exists u)(A_{\text{Smallest}}(u)) \};
\end{aligned}$$

### 5.3 Genericity for Time-Dependent Transformations

Finally, we study notions of genericity determined by groups of time-dependent transformations. Here, we only show results for the groups of arbitrary time-dependent transformations  $\mathcal{F}_{st}$ . We concentrate on the groups  $(\mathcal{A}_{st}, \mathcal{A}_t)$ . The other time-dependent transformation groups will be addressed afterwards (Corollary 5.11). For the groups  $\mathcal{F}_{st}^f$  the problem of identifying sound and complete languages is open, we will discuss the problems concerning this at the end of this section.

We introduce some definitions first. Recall that we introduced, in Section 3.1, the abbreviation  $f(R^{\mathcal{F}})$  for the formula  $\{(f(\mathbf{a}_1, \tau_1), f(\mathbf{a}_2, \tau_2), \dots, f(\mathbf{a}_k, \tau_k)) \mid (\mathbf{a}_1, \tau_1, \mathbf{a}_2, \tau_2, \dots, \mathbf{a}_k, \tau_k) \in R^{\mathcal{F}}\}$ , where  $R$  is a relation name and  $\mathcal{F}$  a spatio-temporal database over a schema  $\sigma$  that contains  $R$ .

*Definition 5.7.* Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be spatio-temporal databases over the schema  $\sigma = \{R_1, \dots, R_m\}$  with underlying dimension  $n$ . The databases  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are called  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -isomorphic if and only if there exists a  $f = (f_{st}, f_t) \in (\mathcal{F}_{st}, \mathcal{F}_t)$  such that for all  $R_i$  in  $\sigma$ ,  $f(R_i^{\mathcal{F}_1}) = R_i^{\mathcal{F}_2}$ .

Recall that a representation of a spatio-temporal database  $\mathcal{F}$  over a schema  $\sigma = \{R_1, \dots, R_m\}$  is a tuple  $(\varphi_1, \dots, \varphi_m)$  of quantifier-free formulas in  $\text{FO}(+, \times, <, 0, 1)$ , such that  $\varphi_i$  describes  $R_i^{\mathcal{F}}$ .

Assuming some order on the characters or symbols that may appear in a  $\text{FO}(+, \times, <, 0, 1)$ -formulas, we associate with each  $\text{FO}(+, \times, <, 0, 1)$ -formula its unique Gödel number (see the encoding step in Section A.4 for details).

These Gödel numbers induce an order on the set of  $\text{FO}(+, \times, <, 0, 1)$ -formulas. Similarly, we can also order tuples of  $\text{FO}(+, \times, <, 0, 1)$ -formulas.

*Definition 5.8.* The  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -*canonization* of a spatio-temporal database  $\mathcal{F}$  over a schema  $\sigma = \{R_1, \dots, R_m\}$ , denoted by  $\mathbf{Canon}_{(\mathcal{F}_{st}, \mathcal{F}_t)}(\mathcal{F})$ , is the spatio-temporal database  $\mathcal{F}'$ , which is  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -isomorphic to  $\mathcal{F}$  and has a representation by quantifier-free  $\text{FO}(+, \times, <, 0, 1)$ -formulas

$$(\varphi \mathbf{Canon}_{(\mathcal{F}_{st}, \mathcal{F}_t)}(R_1), \dots, \varphi \mathbf{Canon}_{(\mathcal{F}_{st}, \mathcal{F}_t)}(R_m))$$

that occurs first, in the above mentioned order on tuples of formulas, among the representations of spatio-temporal databases  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -isomorphic to  $\mathcal{F}$ .

*Definition 5.9.* Let  $\mathcal{F}$  be a spatio-temporal database. The  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -*type* of  $\mathcal{F}$ , denoted  $\text{Type}_{(\mathcal{F}_{st}, \mathcal{F}_t)}(\mathcal{F})$ , equals

$$\{f \in (\mathcal{F}_{st}, \mathcal{F}_t) \mid f(\mathcal{F}) = \mathbf{Canon}_{(\mathcal{F}_{st}, \mathcal{F}_t)}(\mathcal{F})\}.$$

We can derive directly from a similar proposition of Gyssens et al. [1999] that, for a spatio-temporal database  $\mathcal{F}$ , a representation of  $\mathbf{Canon}_{(\mathcal{F}_{st}, \mathcal{F}_t)}(\mathcal{F})$  can be computed if and only if  $(\mathcal{F}_{st}, \mathcal{F}_t)$  is a *semi-algebraic transformation group*.

A transformation group  $\mathcal{G}$  of  $\mathbf{R}^n \times \mathbf{R}$  is semi-algebraic if and only if there exists a semi-algebraic subset of  $\mathbf{R}^l$ , described by a  $\text{FO}(+, \times, <, 0, 1)$ -formula  $\varphi_{\mathcal{G}}$ , for some fixed  $l$ , representing all elements of  $\mathcal{G}$ , such that the set

$$\{(g_1, \dots, g_l, x_1, \dots, x_n, t, x'_1, \dots, x'_n, t') \mid \varphi_{\mathcal{G}}(g_1, \dots, g_l) \wedge \varphi_{\mathcal{G}\text{-img}}(g_1, \dots, g_l, x_1, \dots, x_n, t, x'_1, \dots, x'_n, t')\},$$

also called the *graph* of  $\mathcal{G}$ , is a semi algebraic subset of  $\mathbf{R}^{l+2(n+1)}$ . The formula  $\varphi_{\mathcal{G}\text{-img}}$  expresses that, for the element of  $\mathcal{G}$  represented by the tuple  $(g_1, \dots, g_l)$ , the tuple  $(x_1, \dots, x_n, t)$  is mapped to  $(x'_1, \dots, x'_n, t')$ . It is an easy exercise to show that all groups considered in this article are semi-algebraic.

We now prove the main theorem of this section. The proof technique used here was introduced by Gyssens et al. [1999]. We first sketch the proof technique, but only give details about the aspects of the proof that need modifications in the context of spatio-temporal databases. These modifications are based on proof techniques introduced in Section 4.

**THEOREM 5.10.** *Let  $\sigma$  be a spatio-temporal database schema. The point language  $\text{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{\text{st}}\}, \sigma) + \text{while}$  is sound and complete for the  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic computable spatio-temporal queries over  $\bar{\sigma}$ .*

**PROOF.** We refer to Section A.4 in the appendix for the proof.  $\square$

The reason that the problem of identifying sound and complete languages for the groups  $\mathcal{F}_{st}^f$  is still open, is that for those groups, there is no first-order logic formula expressing their graph. Indeed, it is not possible to express that there should exist a finite number of time moments for which there is a different affinity, when describing the groups  $\mathcal{F}_{st}^f$ . Hence, we cannot use the above proof technique.

The previous theorem has the following corollary:

**COROLLARY 5.11.** *Let  $\sigma$  be a database schema. Let  $(\mathcal{F}_{st}, \mathcal{F}_t)$  be one of the groups  $(\mathcal{A}_{st}, \mathcal{A}_t)$ ,  $(\mathcal{A}_{st}, \mathcal{I}_t)$ ,  $(\mathcal{A}_{st}, Id_t)$ ,  $(\mathcal{S}_{st}, \mathcal{F}_t)$ ,  $(\mathcal{I}_{st}, \mathcal{F}_t)$ , or  $(\mathcal{T}_{st}, \mathcal{F}_t)$  with  $\mathcal{F}_t \in \{\mathcal{A}_t, \mathcal{T}_t, Id_t\}$  and let  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  be as in Table III. The point language  $\text{FO}(\Pi(\mathcal{F}_{st}, \mathcal{F}_t), \sigma) + \text{while}$  is sound and complete for the  $(\mathcal{F}_{st}, \mathcal{F}_t)$ -generic computable spatio-temporal queries over  $\bar{\sigma}$ .*

**PROOF.** The proof of this corollary is similar to the proof of Theorem 5.10. The encoding and decoding programs for the various transformation groups only differ where the transformation in  $\text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}$  is described, and where a coordinate system needs to be defined. The rest of the proof is the same, regardless of the transformation groups considered. The descriptions of the coordinate systems for the various transformation groups can be found in the proof of Corollary 4.20  $\square$

## 6. CONCLUSION AND DISCUSSION

We have investigated different genericity classes relative to the constraint database model for spatio-temporal databases and we have identified sound and complete languages for the  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ , respectively the computable, queries in (most of) these genericity classes. Some results were obtained by techniques introduced by Gyssens et al. [1999], but for time-dependent transformations we have introduced new proof techniques.

For what concerns computationally complete languages these techniques seem to be insufficient to deal with the genericity notions that are expressed by the groups  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ ,  $(\mathcal{A}_{st}^f, \mathcal{I}_t)$ ,  $(\mathcal{A}_{st}^f, Id_t)$ ,  $(\mathcal{S}_{st}^f, \mathcal{F}_t)$ ,  $(\mathcal{I}_{st}^f, \mathcal{F}_t)$ , and  $(\mathcal{T}_{st}^f, \mathcal{F}_t)$  with  $\mathcal{F}_t \in \{\mathcal{A}_t, \mathcal{T}_t, Id_t\}$ . The problem in adapting the proof technique of Theorem 5.10 to these groups is that it is not clear how we can express in the respective point-based logics that two spatio-temporal databases can be mapped to each other by some piece-wise constant affinity. Indeed, since the number of pieces is not defined *a priori*, this might not be expressible. This would imply that yet another new proof technique would be required to deal with the remaining cases.

## APPENDIX

### A.1 Proof of Property 4.6

First, remark that, if we fix  $\mathcal{F}_t$  to be one of  $\{\mathcal{A}_t, \mathcal{T}_t, Id_t\}$ , then

$$(\mathcal{T}_{st}^c, \mathcal{F}_t) \subset (\mathcal{I}_{st}^c, \mathcal{F}_t) \subset (\mathcal{S}_{st}^c, \mathcal{F}_t) \subset (\mathcal{A}_{st}^c, \mathcal{F}_t).$$

Also, if we fix  $\mathcal{F}_{st}$  to be one of  $\{\mathcal{A}_{st}^c, \mathcal{S}_{st}^c, \mathcal{I}_{st}^c, \mathcal{T}_{st}^c\}$ , then

$$(\mathcal{F}_{st}, Id_t) \subset (\mathcal{F}_{st}, \mathcal{T}_t) \subset (\mathcal{F}_{st}, \mathcal{A}_t).$$

Also, all groups  $(\mathcal{F}_{st}, \mathcal{F}_t)$  are subgroups of the affinities of  $\mathbf{R}^n \times \mathbf{R}$ . As we already remarked, if a point predicate is invariant for a certain transformation group  $(\mathcal{F}_{st}, \mathcal{F}_t)$ , it is also invariant for all subgroups of  $(\mathcal{F}_{st}, \mathcal{F}_t)$ .

We now prove invariance for each of the predicates in the sets  $\Pi(\mathcal{F}_{st}, \mathcal{F}_t)$  of Table I.

—The predicate **Between** <sup>$n+1$</sup>  is invariant under elements of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ . It is well known that affinities preserve the betweenness of points. As all groups listed

in Table I are subgroups of the affinities of  $\mathbf{R}^n \times \mathbf{R}$ , the predicate **Between** <sup>$n+1$</sup>  is invariant for all those groups.

- The predicate **Before** is invariant under elements of  $(\mathcal{A}_{st}^c, \mathcal{A}_t)$ , since the elements of  $\mathcal{A}_t$  are monotone bijections of time. As shown in Proposition 3.4, the order on time events is preserved under all strictly monotone increasing bijections of time. The groups  $\mathcal{A}_t, \mathcal{I}_t, Id_t$  are all such bijections.
- The predicate **UnitTime** is invariant under elements of  $(\mathcal{A}_{st}^c, \mathcal{T}_t)$ . It is straightforward that all elements of  $\mathcal{T}_t$ , which are translations in the time direction, preserve the time difference between any two points  $p_1$  and  $p_2$  in  $\mathbf{R}^n \times \mathbf{R}$ .
- The predicates **0<sub>t</sub>** and **1<sub>t</sub>** are invariant under elements of  $(\mathcal{A}_{st}^c, Id_t)$ . It is clear that the identity transformation on the time preserves the fact that a point  $p$  in  $\mathbf{R}^n \times \mathbf{R}$  has time coordinate zero or one.
- The predicate **EqDist** is invariant under elements of  $(\mathcal{S}_{st}^c, \mathcal{A}_t)$ . It is well known that isometries and scalings (and thus similarities) preserve the fact that the distance between one pair of points equals the distance between a second pair of points. The groups  $\mathcal{A}_t, \mathcal{T}_t, Id_t$  all preserve co-temporality of points.
- The predicate **UnitDist** is invariant under elements of  $(\mathcal{I}_{st}^c, \mathcal{A}_t)$ , because isometries are distance preserving transformations.
- The predicates  $\leq_i$  ( $1 \leq i \leq n$ ) are invariant under elements of  $(\mathcal{T}_{st}^c, \mathcal{A}_t)$ . It is easy to verify that if for two points  $p_1$  and  $p_2$  in  $\mathbf{R}^n \times \mathbf{R}$ ,  $\leq_i(p_1, p_2)$  is true for some  $i$  in  $\{1, \dots, n\}$ , also  $\leq_i(f(p_1), f(p_2))$  holds for each  $f$  in  $(\mathcal{T}_{st}^c, \mathcal{F}_t)$ , where  $\mathcal{F}_t$  is one of  $\mathcal{A}_t, \mathcal{T}_t, Id_t$ .
- The predicate **Pos** <sup>$n+1$</sup>  is invariant under elements of  $(\mathcal{T}_{st}^c, \mathcal{A}_t)$ , since translations are orientation-preserving transformations.

This concludes the proof of Property 4.6.

## A.2 Proof of Lemma 4.16

Soundness is proved in two steps. First, we show that every  $\text{FO}(\Pi, \sigma)$ -formula is equivalently expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$  and afterwards that every  $\text{FO}(\Pi, \sigma)$ -formula is invariant under elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ . Both are proved by induction on the structure of  $\text{FO}(\Pi, \sigma)$ -formulas.

Every  $\text{FO}(\Pi, \sigma)$ -formula is expressible in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ . The atomic formulas of  $\text{FO}(\Pi, \sigma)$  are equality on point variables, the predicates **Between** <sup>$n$</sup> , **Before**, **EqCR**<sup>st</sup> and formulas of the type  $R(v_1, \dots, v_l)$ , where  $R$  is a relation name from  $\sigma$ , with arity  $l$ . We now describe, for each of the above types of atomic formulas, how they can be translated into  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$ . A point variable  $v$  occurring in a  $\text{FO}(\Pi, \sigma)$ -formula is translated into real variables  $x_1^v, \dots, x_n^v, t^v$ . Equality between two point variables is then expressed in  $\text{FO}(+, \times, <, 0, 1, \bar{\sigma})$  by requiring that all corresponding coordinates of the two point variables are equal.

We already know that the predicate **Before** is expressible in  $\text{FO}(+, \times, <, 0, 1)$ . The predicate **Between** <sup>$n$</sup>  is translated in a similar way as **Between** <sup>$n+1$</sup> , with the additional restriction that the time coordinates of the variables should be the same.

The formula  $\mathbf{EqCR}^{\text{st}}(u_1, u_2, u_3, v_1, v_2, v_3)$  is translated as the conjunction of the translation of the expression  $\mathbf{Collinear}^n(u_1, u_2, u_3)$ , which is equal to

$$\mathbf{Between}^n(u_1, u_2, u_3) \vee \mathbf{Between}^n(u_2, u_1, u_3) \vee \mathbf{Between}^n(u_1, u_3, u_2)$$

and the formula

$$(t^{v_3} - t^{v_1})^2 \sum_{i=1}^n (x_i^{u_1} - x_i^{u_2})^2 = (t^{v_2} - t^{v_1})^2 \sum_{i=1}^n (x_i^{u_1} - x_i^{u_3})^2.$$

We translate formulas of the type  $R(v_1, \dots, v_l)$ , where  $R$  is a relation name from  $\sigma$  with arity  $l$ , by the formula  $\overline{R}(x_1^{v_1}, \dots, x_n^{v_1}, t^{v_1}, \dots, x_1^{v_l}, \dots, x_n^{v_l}, t^{v_l})$ .

Compositions of atomic formulas by logical connectives and quantifiers are translated in a natural way.

*Every FO( $\Pi, \sigma$ ) formula is invariant for elements of the group  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ .* The only non-trivial part here is showing that all point predicates are  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -invariant. The predicate **Before** is invariant for all transformations  $f = (f_1, f_2, \dots, f_n, f_t)$ , that map  $\mathbf{R}^n \times \mathbf{R}$  to  $\mathbf{R}^n \times \mathbf{R}$ , such that  $f_t$  is a strictly monotone increasing bijection of  $t$  alone (Proposition 3.4). Since all elements of  $\mathcal{A}_t$  are such bijections, this condition is satisfied for  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ . It is well known that affinities preserve the cross-ratio of three points. Because the predicate  $\mathbf{Between}^n$  requires its parameters to be co-temporal (which is preserved by elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ ), these co-temporal points will be transformed by the same affinity and hence their cross-ratio is preserved. Also the predicate  $\mathbf{EqCR}^{\text{st}}$  is invariant under elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ , because the group  $\mathcal{A}_{st}^f$  preserves the cross-ratio between the spatial coordinates of co-temporal points and the group  $\mathcal{A}_t$  preserves the cross-ratio between time coordinates.

This concludes the proof of Lemma 4.16.

### A.3 Proof of Lemma 4.17

Let  $\overline{\psi}$  be a FO( $+, \times, <, 0, 1, \overline{\sigma}$ )-formula. We assume that  $\overline{\psi}$  is in prenex normal form. We now describe the translation of  $\overline{\psi}$  into a formula  $\psi$  of FO( $\Pi, \sigma$ ) (by induction on its structure). In this translation, first the quantifier-free part of  $\overline{\psi}$  is translated and the quantifiers are later added in the obvious way.

To start with, a 2-dimensional ‘‘computation plane’’ is chosen that is used to simulate real variables, constants and all the polynomial equations, polynomial equalities and inequalities.

*The choice of a computation plane.* First of all, two moments in time  $u_{t_0}$  and  $u_{t_E}$  (time moments are simulated in  $\psi$  by variables in  $\mathbf{R}^n \times \mathbf{R}$ ) are chosen such that  $\neg \mathbf{Before}(u_{t_E}, u_{t_0})$ . They form a temporal coordinate system; the formula describing this is as follows:

$$\mathbf{TCoSys}_{\mathcal{A}_t}(u_1, u_2) := \neg \mathbf{Before}(u_2, u_1).$$

Next, in the hyperplane of points co-temporal with  $u_{t_0}$ ,  $n+1$  points  $u_{0,0}, u_{0,1}, \dots, u_{0,n}$  are chosen such that they form an affine coordinate system for the hyperplane co-temporal with  $u_{t_0}$ . The predicate  $\mathbf{CoSys}_{\mathcal{A}}^n$ , expressing this, is similar to the previously introduced predicate  $\mathbf{CoSys}_{\mathcal{A}}$  (see the proof of Theorem 4.10),



except that some constraints are added that express that the points should be co-temporal.

As the variables  $u_{t_O}, u_{t_E}, u_{0,0}, u_{0,1}, \dots, u_{0,n}$  represent arbitrary points (up to the mentioned restrictions), they parameterise the translation of  $\bar{\psi}$ . To start with,  $\psi$  will contain the subformula  $\psi_{\text{comp}}$ , defined as

$$\begin{aligned} \psi_{\text{comp}}(u_{t_O}, u_{t_E}, u_{0,0}, u_{0,1}, \dots, u_{0,n}) := & \mathbf{TCoSys}_A(u_{t_O}, u_{t_E}) \\ & \wedge \mathbf{CoSys}_A^n(u_{0,0}, u_{0,1}, \dots, u_{0,n}) \wedge \mathbf{Cotemp}(u_{t_O}, u_{0,0}), \end{aligned}$$

as a conjunct.

We will use the 2-dimensional plane through the points  $u_{0,0}, u_{0,1}$  and  $u_{0,2}$  as a “computation plane”. The idea is that we will simulate real variables and constants by points on the line through  $u_{0,0}$  and  $u_{0,1}$  and that addition and multiplication of real terms are simulated by FO( $\Pi$ ) expressions in the plane through  $u_{0,0}, u_{0,1}$  and  $u_{0,2}$ .

- *The translation of terms and atomic formulas.* A quantifier-free FO( $+, \times, <, 0, 1, \bar{\sigma}$ )-formula may contain the following terms and atomic subformulas: real variables; the constants 0 and 1; polynomial constraints; and relation predicates where the relation names from  $\bar{\sigma}$  are used. We translate each separately.
- *The translation of real variables.* Each real variable  $x$  appearing in the formula  $\bar{\psi}$  is translated into a spatio-temporal variable  $v$ . Also,  $\psi$  will contain a conjunct

$$\psi_{\text{var}}(v) := \mathbf{Collinear}^n(u_{0,0}, u_{0,1}, v),$$

expressing that  $v$  is in the computation plane on the line connecting  $u_{0,0}$  and  $u_{0,1}$ . The idea is that a real variable  $x$  taking concrete value  $a$ , is simulated by requiring that  $v$  is such that  $\frac{u_{0,0}b}{u_{0,0}u_{0,1}}$  equals  $a$ .

- *The translation of the constants 0 and 1.* The real constants 0 and 1 that may appear in  $\bar{\psi}$  are translated into  $u_{0,0}$  and  $u_{0,1}$  respectively.
- *The translation of polynomial constraints.* The arithmetic operations (addition and multiplication) on real terms will be simulated in the computation plane ( $u_{0,0}, u_{0,1}, u_{0,2}$ ). It was shown by Tarski [Schwabhäuser et al. 1983] (the results of Tarski were also used in Gyssens et al. [1999]) that all arithmetic operations on points that are located on the line through  $u_{0,0}$  and  $u_{0,1}$  can be simulated in the plane ( $u_{0,0}, u_{0,1}, u_{0,2}$ ) using only the construct **Between**<sup>*n*</sup>. Hence, a subformula  $p(x_1, \dots, x_m) > 0$ , with  $p$  a polynomial with integer coefficients, using the translation of the real variables  $x_1, \dots, x_m$  in point variables  $v_1, \dots, v_m$ , is translated into  $\psi_{\text{poly}}(u_{0,0}, u_{0,1}, u_{0,2}, v_1, \dots, v_m)$ , defined using the predicate **Between**<sup>*n*</sup>.

The correctness of the three above translations can be demonstrated as that of the similar translations in Gyssens et al. [1999].

- *The translation of relation predicates.* A subformula of  $\bar{\psi}$  of type  $\bar{R}(x_{1,1}, \dots, x_{1,n}, x_{1,t}, \dots, x_{m,1}, \dots, x_{m,n}, x_{m,t})$ , where  $\bar{R} \in \bar{\sigma}$  and where  $m$  is the arity of  $\bar{R}$

in  $\sigma$ , is translated into a formula

$$R(v_1, \dots, v_m)$$

and  $\psi$  has a conjunct expressing that the point variables  $v_{1,1}, \dots, v_{1,n}, v_{1,t}, \dots, v_{m,1}, \dots, v_{m,n}, v_{m,t}$ , that are the translations of  $x_{1,1}, \dots, x_{1,n}, x_{1,t}, \dots, x_{m,1}, \dots, x_{m,n}, x_{m,t}$ , are the coordinates of  $v_1, \dots, v_m$  respectively. For the moment, we assume that the variables  $x_{i,t}$  and  $x_{j,t}$  are different for  $1 \leq i < j \leq m$  and later show how to deal with the general case. Indeed, recall that each variable  $x_{i,j}$  ( $1 \leq i \leq m, 1 \leq j \leq n$ ) and  $x_{i,t}$  ( $1 \leq i \leq m$ ) is already translated into a point variable  $v_{i,j}$  and  $v_{i,t}$ , which are all collinear with  $u_{0,0}$  and  $u_{0,1}$ . To express the link between the coordinates of point variables  $v_1, \dots, v_m$  and the point variables  $v_{i,j}$  and  $v_{i,t}$ , we proceed as follows. We associate with each point variable  $v_i$  ( $1 \leq i \leq m$ ) the following set of point variables:

- (i)  $n + 1$  point variables  $u_{i,0}, \dots, u_{i,n}$  representing an  $n$ -dimensional coordinate system which is co-temporal with  $v_i$ ; and
- (ii)  $n$  point variables  $v'_{i,j}$  which are collinear with  $u_{i,0}$  and  $u_{i,1}$ , such that  $v'_{i,j}$  represents the  $j$ th coordinate of  $v_i$  with respect to the coordinate systems specified by  $u_{i,0}, \dots, u_{i,n}$ , and such that the coordinate of  $v'_{i,j}$ , on the line through  $u_{i,0}$  and  $u_{i,1}$ , gives the same cross ratio with respect to these points as the coordinate of  $v_{i,j}$ , on the line through  $u_{0,0}$  and  $u_{0,1}$ , gives with respect to these points, that is,  $\frac{\overrightarrow{u_{i,0}v'_{i,j}}}{\overrightarrow{u_{i,0}u_{i,1}}} = \frac{\overrightarrow{u_{0,0}v_{i,j}}}{\overrightarrow{u_{0,0}u_{0,1}}}$ .

As explained before, the first set of  $n + 1$  point variables can be defined using the expression

$$\mathbf{CoSys}_A^n(u_{i,0}, u_{i,1}, \dots, u_{i,n}) \wedge \mathbf{Cotemp}(u_{i,0}, v_i).$$

For the second set of  $n$  point variables, we first observe that from Gyssens et al. [1999], we know that we can express, using **Between** <sup>$n$</sup> , that  $n$  point variables  $v'_{i,1}, \dots, v'_{i,n}$  represent the spatial coordinates of the point variable  $v_i$  relative to a chosen spatial coordinate system (in this case, the coordinate system specified by  $u_{i,0}, \dots, u_{i,n}$ ). In order to establish the link between the point variables  $v'_{i,j}$  in the plane specified by  $u_{i,0}, \dots, u_{i,n}$  and the point variables  $v_{i,j}$  in the computation plane we need to use the predicate **EqCR**<sup>s</sup>. The predicate **EqCR**<sup>s</sup> performs a transformation between the affine coordinate systems at two different time moments, and so connects each  $v'_{i,j}$  to a  $v_{i,j}$  ( $i = 1, \dots, m, j = 1, \dots, n$ ). Remark that all  $v'_{i,j}$  are collinear with  $u_{i,0}$  and  $u_{i,1}$ , and that all  $v_{i,j}$  are collinear with  $u_{0,0}$  and  $u_{0,1}$ . Therefore, **EqCR**<sup>s</sup> can be used to express this equality of cross ratios.

Until now, we only considered the spatial coordinates. To link the temporal variables  $v_{i,t}$  to the temporal coordinate of  $v_i$ , we use the expression **EqCR**<sup>st</sup>( $u_{0,0}, u_{0,1}, v_{i,t}, u_{t_0}, u_{t_E}, v_i$ ). Recall that the predicate **EqCR**<sup>st</sup> can be used to relate the cross ratio of points on the time axis to the cross ratio of points, representing coordinates on the line through  $u_{0,0}$  and  $u_{0,1}$ , and thus connects each  $v_i$  to a  $v_{i,t}$  ( $i = 1, \dots, m$ ).

Putting everything together results in the expression  $\psi_{\text{rel}}$ :

$$\begin{aligned}
& (\exists v_1)(\exists v_2) \dots (\exists v_m) \left( R(v_1, v_2, \dots, v_m) \wedge \bigwedge_{i=1}^m \mathbf{CoSys}_{\mathcal{A}}^n(u_{i,0}, u_{i,1}, \dots, u_{i,n}) \wedge \right. \\
& \quad [-2pt] \bigwedge_{i=1}^m \mathbf{Cotemp}(u_{i,0}, v_i) \wedge (\exists v'_{1,1}) \dots (\exists v'_{1,n}) \dots (\exists v'_{m,1}) \dots (\exists v'_{m,n}) \\
& \quad \left( \bigwedge_{i=1}^m \mathbf{Coordinates}^n(u_{i,0}, u_{i,1}, \dots, u_{i,n}, v'_{i,1}, \dots, v'_{i,n}, v_i) \wedge \right. \\
& \quad \quad \bigwedge_{i=1}^m \bigwedge_{j=1}^n \mathbf{EqCR}^s(u_{0,0}, u_{0,1}, v_{i,j}, u_{i,0}, u_{i,1}, v'_{i,j}) \wedge \\
& \quad \quad \left. \left. \bigwedge_{i=1}^m \mathbf{EqCR}^{\text{st}}(u_{0,0}, u_{0,1}, v_{i,t}, u_{t_0}, u_{t_E}, v_i) \right) \right)
\end{aligned}$$

where  $\mathbf{Coordinates}^n(u_{i,0}, \dots, u_{i,n}, v'_{i,1}, \dots, v'_{i,n}, v_i)$  expresses for each  $(1 \leq j \leq n)$  that  $v'_{i,j}$  is represents the  $j$ th coordinate of  $v_i$  with respect to the coordinate systems specified by  $u_{i,0}, \dots, u_{i,n}$ .

We now show the correctness of the above translation of a relation predicate. We have to prove that for each spatio-temporal database  $\mathcal{F}$ , and for any points  $p_{t_0}, p_{t_E}, p_{0,0}, \dots, p_{0,n}, \dots, p_{m,0}, \dots, p_{m,n}, q_{1,1}, \dots, q_{1,n}, q_{1,t}, \dots, q_{m,1}, \dots, q_{m,n}, q_{m,t}$ :

$$(\mathbf{R}^n \times \mathbf{R}, \Pi^{\mathbf{R}^n \times \mathbf{R}}, \mathcal{F}) \models \psi_{\text{rel}}[p_{t_0}, p_{t_E}, p_{0,0}, \dots, p_{0,n}, \dots, p_{m,0}, \dots, p_{m,n}, q_{1,1}, \dots, q_{1,n}, q_{1,t}, \dots, q_{m,1}, \dots, q_{m,n}, q_{m,t}]$$

if and only if

$$(\mathbf{R}, +, \times, 0, 1, \alpha(\overline{\mathcal{F}})) \models \overline{R} \left[ \frac{\overrightarrow{p_{0,0}q_{1,1}}}{\overrightarrow{p_{0,0}p_{0,1}}}, \dots, \frac{\overrightarrow{p_{0,0}q_{1,n}}}{\overrightarrow{p_{0,0}p_{0,1}}}, \frac{\overrightarrow{p_{0,0}q_{1,t}}}{\overrightarrow{p_{0,0}p_{0,1}}}, \dots, \frac{\overrightarrow{p_{0,0}q_{m,1}}}{\overrightarrow{p_{0,0}p_{0,1}}}, \dots, \frac{\overrightarrow{p_{0,0}q_{m,n}}}{\overrightarrow{p_{0,0}p_{0,1}}}, \frac{\overrightarrow{p_{0,0}q_{m,t}}}{\overrightarrow{p_{0,0}p_{0,1}}} \right],$$

where  $\alpha = (\alpha_{st}, \alpha_t) \in (\mathcal{A}_{st}^f, \mathcal{A}_t)$  is the affinity which maps  $(p_{0,0}, \dots, p_{0,n})$  to the spatial standard basis at time  $\tau_0 = 0$ ,  $(p_{i,0}, \dots, p_{i,n})$  to the spatial standard basis at time  $\tau_i = \alpha(\tau_{p_{i,0}})$ , where  $\alpha_t$  is uniquely determined on the time axis by  $\alpha_t(\tau_{p_0}) = 0$  and  $\alpha_t(p_{t_E}) = 1$ . Note that by assumption,  $x_{i,t} \neq x_{j,t}$  for  $(1 \leq i < j < m)$  and hence also  $\tau_{p_{i,0}}$  and  $\tau_{p_{j,0}}$ , and consequently  $\tau_i \neq \tau_j$  for  $(1 \leq i < j < m)$ . This condition is essential to ensure that  $\alpha_t$  exists and is well defined. Indeed, suppose that there exists an  $i$  and  $j$  such that  $\tau_{p_{i,0}} = \tau_{p_{j,0}}$  and hence  $\tau_i = \tau_j$ . Then we would require that  $\alpha$  maps two possibly different co-temporal coordinate systems  $(p_{i,0}, \dots, p_{i,n})$  and  $(p_{j,0}, \dots, p_{j,n})$  the same standard basis. This can clearly not be done by a  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ -generic query.

We know that the formula  $\psi_{\text{rel}}$  is true for the points  $p_{t_0}, p_{t_E}, p_{0,0}, \dots, p_{0,n}, \dots, p_{m,0}, \dots, p_{m,n}, q_{1,1}, \dots, q_{1,n}, q_{1,t}, \dots, q_{m,1}, \dots, q_{m,n}, q_{m,t}$  if and only if there exist points  $p_1, \dots, p_m, q'_{1,1}, \dots, q'_{1,n}, \dots, q'_{m,1}, \dots, q'_{m,n}$  such that for each

$i = 1, \dots, m$ :

$$\overrightarrow{p_{0,0}p_i} = \overrightarrow{p_{i,0}p_i} + \overrightarrow{p_{0,0}p_{i,0}} = \sum_{j=1}^n \frac{\overrightarrow{p_{i,0}q_{i,j}}}{\overrightarrow{p_{i,0}p_{i,1}}} \overrightarrow{p_{i,0}p_{i,j}} + \overrightarrow{p_{0,0}p_{i,0}}, \quad (1)$$

and the following equations hold:

$$\frac{\overrightarrow{p_{i,0}q_{i,j}}}{\overrightarrow{p_{i,0}p_{i,1}}} = \frac{\overrightarrow{p_{0,0}q_{i,j}}}{\overrightarrow{p_{0,0}p_{0,1}}}, \quad 1 \leq j \leq n, \quad (2)$$

$$\frac{\tau_{p_i} - \tau_{p_{i,0}}}{\tau_{p_{i,E}} - \tau_{p_{i,0}}} = \frac{\overrightarrow{p_{0,0}q_{i,t}}}{\overrightarrow{p_{0,0}p_{0,1}}}. \quad (3)$$

Using Eq. (2), Eq. (1) is equivalent to

$$\overrightarrow{p_{0,0}p_i} = \sum_{j=1}^n \frac{\overrightarrow{p_{0,0}q_{i,j}}}{\overrightarrow{p_{0,0}p_{0,1}}} \overrightarrow{p_{i,0}p_{i,j}} + \overrightarrow{p_{0,0}p_{i,0}}. \quad (4)$$

Considering the fact that  $\alpha$  is a linear transformation, and using Eq. (4), the following holds:

$$\alpha(\overrightarrow{p_{0,0}p_i}) = \sum_{j=1}^n \frac{\overrightarrow{p_{0,0}q_{i,j}}}{\overrightarrow{p_{0,0}p_{0,1}}} \alpha(\overrightarrow{p_{i,0}p_{i,j}}) + \alpha(\overrightarrow{p_{0,0}p_{i,0}}).$$

Moreover, let  $e_i(\tau)$  be the  $i$ th vector of the standard spatial basis at time  $\tau$  and denote by  $e_i = e_i(0)$ . We then have

$$\alpha(\overrightarrow{p_{0,0}p_i}) = \sum_{j=1}^n \frac{\overrightarrow{p_{0,0}q_{i,j}}}{\overrightarrow{p_{0,0}p_{0,1}}} \overrightarrow{e_0(\tau_i)e_j(\tau_i)} + \overrightarrow{e_0e_0(\tau_i)}.$$

As Eq. (3) is invariant under elements of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$ , we also have that

$$\frac{\alpha(\tau_{p_i}) - \alpha(\tau_{p_{i,0}})}{\alpha(\tau_{p_{i,E}}) - \alpha(\tau_{p_{i,0}})} = \frac{\tau_i - 0}{1 - 0} = \tau_i = \frac{\overrightarrow{p_{0,0}q_{i,t}}}{\overrightarrow{p_{0,0}p_{0,1}}}.$$

So we have that:

$$\alpha(\overrightarrow{p_{0,0}p_i}) = \sum_{j=1}^n \frac{\overrightarrow{p_{0,0}q_{i,j}}}{\overrightarrow{p_{0,0}p_{0,1}}} \overrightarrow{e_0(\tau_i)e_j(\tau_i)} + \frac{\overrightarrow{p_{0,0}q_{i,t}}}{\overrightarrow{p_{0,0}p_{0,1}}} \overrightarrow{e_0e_{n+1}}.$$

Since all standard bases  $(e_0(\tau_i), \dots, e_n(\tau_i))$  are parallel along the time axis, we have that

$$\alpha(\overrightarrow{p_{0,0}p_i}) = \sum_{j=1}^n \frac{\overrightarrow{p_{0,0}q_{i,j}}}{\overrightarrow{p_{0,0}p_{0,1}}} \overrightarrow{e_0e_j} + \frac{\overrightarrow{p_{0,0}q_{i,t}}}{\overrightarrow{p_{0,0}p_{0,1}}} \overrightarrow{e_0e_{n+1}}.$$

This completes the correctness proof for the conversion of relational predicates.

- The translation of composed formulas.* When all the atomic subformulas of  $\overline{\psi}$  have been translated as described above, the logical connectives can be added in a natural way. We assume that two atomic formulas  $\overline{\chi}_1$  and  $\overline{\chi}_2$  are translated already, into  $\chi_1$  and  $\chi_2$ . The translations of  $\overline{\chi}_1 \wedge \overline{\chi}_2$  and  $\overline{\chi}_1 \vee \overline{\chi}_2$  are  $\chi_1 \wedge \chi_2$  and  $\chi_1 \vee \chi_2$ , respectively. The formula  $\neg \overline{\chi}_1$  is translated into  $\neg \chi_1$ . Remember that with the conversion of a formula  $\overline{\psi}_{\text{rel}}$  of type  $\overline{R}(x_{1,1}, \dots, x_{1,n}, x_{1,t}, \dots, x_{m,1}, \dots, x_{m,n}, x_{m,t})$  we assumed that  $x_{i,t} \neq x_{j,t}$  for any  $(1 \leq i < j < m)$ . The reason is that we want to have only one affine coordinate system for every different time moment considered in that formula. Indeed, an element  $\alpha$  of  $(\mathcal{A}_{st}^f, \mathcal{A}_t)$  is a one-to-one mapping from the snapshots of a certain input database  $\mathcal{F}$  to the snapshots of the output database  $\alpha(\mathcal{F})$ . Therefore, we cannot map two different co-temporal coordinate systems to the same standard coordinate system using such an affinity.
- Suppose now that  $x_{i,t} = x_{j,t}$  for some  $(1 \leq i < j \leq m)$ . Then we adapt the previous translation with the extra requirement that  $v_{i,k} = v_{j,k}$  for  $k = 0, \dots, n$  and we have unique coordinate system for each point occurring in time.
- When translating an FO(+, ×, <, 0, 1,  $\overline{\sigma}$ )-formula  $\overline{\psi}$ , it is in general not known in advance which time coordinates are equal (this may depend on the input database; and it is undecidable in general which time coordinates are equal in an FO(+, ×, <, 0, 1,  $\overline{\sigma}$ )-formula). To circumvent this problem, we consider all possible orders (using **Before**) of the time variables of  $\overline{\psi}$  (a real variable denoting a time moment is recognized as it appears on the  $i(n+1)$ -th place ( $i = 1, \dots, m$ ) in the argument list of a spatio-temporal relation predicate) and take the disjunction over all possible orders of these time variables. We denote the set of all possible orders by  $P$ .
- For each  $\rho \in P$  the formula  $\psi_\rho$  is the translation of  $\overline{\psi}$  taken the (in)equalities into account according to the order of the time variables corresponding to  $\rho$ . Hence, each  $\psi_\rho$  formula can have a different number  $\ell_\rho$  of free variables, depending on  $\rho$ . We denote by  $\ell$  the total number of free variables across all formulas  $\psi_\rho, \rho \in P$ .
- When connecting several subformulas, the same principle has to be used, as arithmetic subformulas can impose equality on different time variables.
- When applying the thus obtained translation of the quantifier-free part of  $\psi$  to a spatio-temporal database instance, only some of the disjuncts will apply (possibly depending on the particular input database).
- Formulas with quantifiers.* Finally, the quantifier prefix of  $\overline{\psi}$  is translated in the natural way. Suppose that we already translated the quantifier-free formula  $\overline{\chi}$  into the formula  $\chi$ . Then the translation of  $(\exists x)(\overline{\chi})$  is  $(\exists v)(\chi)$ , where  $v$  is the point variable associated to  $x$  for which we have already declared **Collinear** <sup>$n$</sup>  $(u_{0,0}, u_{0,1}, v)$ . This concludes the proof of Lemma 4.17.

#### A.4 Proof of Theorem 5.10

It suffices to show that an  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic computable query  $Q$  over  $\overline{\sigma}$  can be simulated in the language FO(**{Between** <sup>$n$</sup> , **Before**, **EqCR**<sup>st</sup>},  $\sigma$ ) + while. We first briefly sketch the proof strategy, including the conversion procedure and

the encoding and decoding step, that appear in it. Later the coding and decoding will be explained in more detail. For the remainder of this proof,  $\Pi$  will denote the set **{Between<sup>n</sup>, Before, EqCR<sup>st</sup>}**.

We start with the encoding that will be used to convert formulas that represent spatio-temporal relations into natural numbers.

*The encoding mechanism.* Let  $\mathcal{F}$  be a spatio-temporal database over  $\sigma$ . Let  $K$  be the maximum of the arities of all relations in  $\sigma$  and the query  $Q$ . Let  $n$  be the underlying dimension. Then each relation of  $\mathcal{F}$  can be represented by a quantifier-free  $\text{FO}(+, \times, <, 0, 1)$ -formula using only the variables  $x_1, \dots, x_{(n+1)K}$ , the symbols  $\leq, +, \times, (, ), \vee$  and  $\neg$ , and the constants 0 and 1.

We denote these  $9 + (n + 1)K$  symbols by  $s_1, \dots, s_{9+(n+1)K}$ . Hence, we can encode a quantifier-free  $\text{FO}(+, \times, <, 0, 1)$ -formula as a string  $s = s_{i_1} \dots s_{i_k}$  as the natural number  $N = p_1^{i_1} \dots p_k^{i_k}$ , where  $p_j$  is the  $j$ -th prime number. And we denote  $N$  by  $\text{Encode}(s)$ .

*Proof strategy.* Given a spatio-temporal database  $\mathcal{F}$  over a schema  $\sigma = \{R_1, \dots, R_m\}$ , the simulation of a  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic  $k$ -ary computable query on input  $\mathcal{F}$  is broken up into three steps:

- The encoding step.* The database  $\mathcal{F}$  is encoded as a tuple of natural numbers  $(N_{R_1}, \dots, N_{R_m})$ , one for each relation of the database. Here,  $N_{R_i} = \text{Encode}(s_i)$ , where  $(s_1, \dots, s_m)$  are the string representation of the quantifier-free formulas  $\varphi_{\text{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R_i)}$  ( $i = 1, \dots, m$ ) of the database  $\text{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$ . It will be shown below that this encoding can be performed in the language  $\text{FO}(\Pi, \sigma) + \text{while}$ . The set  $\text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$  is also computed, to be used in the decoding step.
- The computing step.* It was shown that query the language  $\text{FO}(\Pi) + \text{while}$  has full computational power on the natural numbers, by simulating a counter machine [Gyssens et al. 1999]. More specifically, one can simulate a counter machine  $M$  in  $\text{FO}(\Pi) + \text{while}$  such that on input  $(N_{R_1}, \dots, N_{R_m})$ ,  $M$  halts if and only if  $Q$  is defined on the corresponding  $\mathcal{F}$  and  $M$  will output a natural number  $N_q$  which is the encoding of  $Q(\mathcal{F})$ .
- The decoding step.* If  $M$  terminates on input  $(N_{R_1}, \dots, N_{R_m})$  then it outputs a natural number  $N_q$ . Using  $\text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$ , the decoding algorithm computes the point set of which  $N_q$  is the encoding. This can be implemented in the language  $\text{FO}(\Pi) + \text{while}$ .

We show next the details in the encoding and decoding algorithms that are different for  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic queries, as compared to the affine-generic queries considered in Gyssens et al. [1999]. For ease of exposition, we will assume for the remainder of this proof that the input spatio-temporal database  $\mathcal{F}$  has only one relation, with arity one, that is,  $\sigma = \{R\}$ . For relations with arity greater than one, the encoding algorithm has to consider more variables. If the input database contains more relations, the encoding has to be generalized to encoding tuples of relations.

*The encoding algorithm can be expressed in  $\text{FO}(\Pi) + \text{while}$ .* Roughly speaking, the encoding procedure enumerates all natural numbers and meanwhile



stores the evaluation of the terms and formulas that are encoded by those numbers in relations that are called  $T$  and  $F$ , respectively. This enumeration continues until one natural number is found that encodes a relation that is  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -isomorphic to  $R$ . This relation, for which the evaluation is stored in  $F$ , corresponds to  $\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R)$ . The set  $Type_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$  is also computed, to use in the decoding step.

First, we explain the role of the relations  $T$  and  $F$  in more detail, as well as the way they are built during the encoding process.

The encoding program builds up terms and formulas until the formula is found that encodes  $\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$ . The terms and formulas are stored in the relations  $T$  and  $F$ . In general, the arity of  $T$  is  $(n + 1) + 2 + 2 + l \times (n + 1)$ , where  $n$  is the underlying dimension and  $l = ar(R)$ . Under the assumption that  $ar(R) = 1$  and the underlying dimension is 2, each tuple in  $T$  is of the form

$$(u_{t_O}, u_{t_E}, u_0, u_1, u_2, u_t, p_1, p_2, p_t, v),$$

where  $(u_{t_O}, u_{t_E})$  is a temporal coordinate system,  $(u_0, u_1, u_2)$  a spatial coordinate system,  $u_t$  the encoding of a term which only uses the variables  $x_1, x_2, x_t$  (which are translated into  $v_1, v_2, v_t$ ), and  $v$  the value of the term when evaluated under the valuation  $v_1 \mapsto p_1, v_2 \mapsto p_2, v_t \mapsto p_t$ . The arity of  $F$  is  $(n+1)+2+1+l \times (n+1)$ . Under the same assumptions, each tuple in  $F$  is of the form

$$(u_{t_O}, u_{t_E}, u_0, u_1, u_2, u_f, p_1, p_2, p_t),$$

where  $(u_{t_O}, u_{t_E})$  and  $(u_0, u_1, u_2)$  are as before,  $u_f$  the encoding of a formula  $\bar{\varphi}$  which only uses the variables  $x_1, x_2, x_t$ , and where  $\varphi(p_1, p_2, p_t)$  is true.

We now give the structure of the encoding program in  $\mathbf{FO}(\{\mathbf{Between}^n, \mathbf{Before}, \mathbf{EqCR}^{st}\}) + \mathbf{while}$ . In this algorithm, it is assumed that substrings  $s'$  of a string  $s$  is encountered in the enumeration before  $s$  is encountered. The input is an  $\mathbf{FO}(+, \times, <, 0, 1, \bar{R})$ -sentence.

```

m := 0;
T := ∅;
F := ∅;
found := False;
while not found do
  m := m + 1;
  if m encodes  $x_1$ 
    T := T ∪ {(utO, utE, u0, u1, u2, m, p1, p2, pt, p1) |
      p1, p2, pt collinear with u0 and u1};
  else if m encodes  $x_2$  then
    T := T ∪ {(utO, utE, u0, u1, u2, m, p1, p2, pt, p2) |
      p1, p2, pt collinear with u0 and u1};
  else if m encodes  $x_t$  then
    T := T ∪ {(utO, utE, u0, u1, u2, m, p1, p2, pt, pt) |
      p1, p2, pt collinear with u0 and u1};
  else if m encodes 0 then
    T := T ∪ {(utO, utE, u0, u1, u2, m, p1, p2, pt, u0) |
      p1, p2, pt collinear with u0 and u1};
  else if m encodes 1 then
    T := T ∪ {(utO, utE, u0, u1, u2, m, p1, p2, pt, u1) |
      p1, p2, pt collinear with u0 and u1};
  else if m encodes (s + t) then
    T := T ∪ {(utO, utE, u0, u1, u2, m, p1, p2, pt, pe) |

```

$$\begin{aligned}
& T(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(s), p_1, p_2, p_t, p_c) \wedge \\
& \quad T(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(t), p_1, p_2, p_t, p_d) \wedge \mathbf{Plus}(p_c, p_d, p_e)); \\
\mathbf{else\ if\ } m \text{ encodes } (s \times t) \mathbf{\ then} \\
& \quad T := T \cup \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, m, p_1, p_2, p_t, p_e) | \\
& \quad \quad T(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(s), p_1, p_2, p_t, p_c) \wedge \\
& \quad \quad T(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(t), p_1, p_2, p_t, p_d) \wedge \mathbf{Times}(p_c, p_d, p_e)\}; \\
\mathbf{else\ if\ } m \text{ encodes } (s \leq t) \mathbf{\ then} \\
& \quad F := F \cup \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, m, p_1, p_2, p_t) | \\
& \quad \quad (\exists c)(\exists d)(T(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(s), p_1, p_2, p_t, p_c) \wedge \\
& \quad \quad T(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(t), p_1, p_2, p_t, p_d) \wedge \mathbf{Less}(p_c, p_d))\}; \\
\mathbf{else\ if\ } m \text{ encodes } (\neg\varphi) \mathbf{\ then} \\
& \quad F := F \cup \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, m, p_1, p_2, p_t) | \\
& \quad \quad \neg F(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(\varphi), p_1, p_2, p_t)\}; \\
\mathbf{else\ if\ } m \text{ encodes } (\varphi \vee \psi) \mathbf{\ then} \\
& \quad F := F \cup \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, m, p_1, p_2, p_t) | \\
& \quad \quad F(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(\varphi), p_1, p_2, p_t) \vee \\
& \quad \quad F(u_{t_O}, u_{t_E}, u_0, u_1, u_2, enc(\psi), p_1, p_2, p_t)\}; \\
\mathbf{end\ if} \\
\mathbf{found} := m \text{ encodes a formula which represents } \mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R); \\
\mathbf{end\ while} \\
N_{\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R)} := m; \\
Type_{(\mathcal{A}_{st}, \mathcal{A}_t)} := \{a \in (\mathcal{A}_{st}, \mathcal{A}_t) \mid a(R) = \mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R)\};
\end{aligned}$$

We will discuss in detail

- (i) the representation of natural numbers (as we only have point-variables),
- (ii) the expression that checks whether a certain natural number encodes a formula which represents  $\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$ , and
- (iii) the computation of the set  $Type_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R)$ .

All other elements of the encoding can be adopted from the proof of Gyssens et al. [1999] with only slight modifications. For ease of exposition, we give the formulas for  $n = 2$ .

- (i) Natural numbers can be represented by  $(n + 1)$ -dimensional points using the computation plane technique introduced in Section 4. Further on, in de encoding and decoding algorithm, we need to simulate assignments such as  $m := 0$  and  $m := m + 1$  (since we have to run through all natural numbers in those algorithms). As an illustration, we explain here how these are simulated in  $\text{FO}(\Pi) + \text{while}$ . The expression  $m := 0$ , for example, is translated in  $\text{FO}(\Pi) + \text{while}$  by assigning to a spatio-temporal relation a point that is the origin of the chosen computation plane. The translated expression is

$$\begin{aligned}
N := \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v) \mid \mathbf{TCoSys}_{\mathcal{A}}(u_{t_O}, u_{t_E}) \wedge \\
\mathbf{CoSys}_{\mathcal{A}}^n(u_0, u_1, u_2) \wedge \mathbf{Collinear}(u_0, u_1, v) \wedge v = u_0\}.
\end{aligned}$$

For the assignment  $m := m + 1$ , we have:

$$\begin{aligned}
N := \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v) \mid (\exists w)(N(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w) \wedge \\
\mathbf{Plus}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w, u_1, v))\}.
\end{aligned}$$

The predicate **Plus**, which expresses that, relative to a computation plane, a certain point represents the sum of two other points can be written in  $\text{FO}(\Pi) + \text{while}$  because of Theorem 4.15.

- (ii) We now give the expression  $\varphi$  that checks whether a certain natural number  $m$  encodes a formula which represents  $\varphi_{\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R_i)}$  ( $i = 1, \dots, m$ ). Remember that the evaluation of the formula encoded by  $m$  is stored in the relation  $F$ . This relation has arity  $n + 3 + K$ , where  $K$  is the maximal arity in the input database schema. Let  $(p_1, p_2, \dots, p_{n+3+K})$  be a tuple of points satisfying  $F$ . The points  $p_1$  and  $p_2$  are a temporal coordinate system,  $p_{n+3}$  represents the natural number  $m$  encoding the formula and  $p_3, p_4, \dots, p_{n+2}$  form a hyperplane of which the plane through  $p_3, p_4$  and  $p_5$  will be used as a computation plane. The last  $K$  points are the translation of the free variables in the formula encoded by  $m$ .

Let the formula  $\psi_{(\mathcal{A}_{st}, \mathcal{A}_t)}$  be the translation of the semi-algebraic description  $\varphi_{(\mathcal{A}_{st}, \mathcal{A}_t)}$  of the group  $(\mathcal{A}_{st}, \mathcal{A}_t)$ .

Intuitively, the next formula checks, for a natural number  $m$ , whether there exists an element of the group  $(\mathcal{A}_{st}, \mathcal{A}_t)$  that maps each point in  $R$  to a point in the set of points satisfying the formula encoded by  $m$ , the evaluation of which is stored in  $F$ .

The following formula  $\psi$  checks whether the right quantifier-free formula has been found. It reflects the stop condition of the while-loop that runs through the natural numbers. This formula  $\psi$  can be written as

$$\begin{aligned} & (\forall u_{t_O})(\forall u_{t_E})(\forall u_0)(\forall u_1)(\forall u_2) \left( (\mathbf{TCoSys}_{\mathcal{A}}(u_{t_O}, u_{t_E}) \wedge \right. \\ & \quad \mathbf{CoSys}_{\mathcal{A}}^n(u_0, u_1, u_2) \rightarrow (\exists v_\alpha)(\exists v_\beta)(\exists w)(\forall u_t)(\exists v_{a_{1,1}})(\exists v_{a_{1,2}})(\exists v_{a_{2,1}})(\exists v_{a_{2,2}}) \\ & \quad (\exists v_{b_1})(\exists v_{b_2})(N(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w) \wedge \\ & \quad (\forall v_x)(\forall v_y)(\forall v_t)(F(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w, v_x, v_y, v_t) \leftrightarrow \\ & \quad (\exists v)(\exists v'_x)(\exists v'_y)(R(v) \wedge \mathbf{comp-coord}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v, v'_x, v'_y, u_t) \wedge \\ & \quad \psi_{(\mathcal{A}_{st}, \mathcal{A}_t)}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_{a_{1,1}}, v_{a_{1,2}}, v_{a_{2,1}}, v_{a_{2,2}}, v_{b_1}, v_{b_2}, u_t, v_\alpha, v_\beta, \\ & \quad \left. \left. v'_x, v'_y, u_t, v_x, v_y, v_t \right) \right) \right). \end{aligned}$$

In the above formula, we omitted, for all point variables except  $v$ , the subformulas expressing collinearity with  $v_0$  and  $v_1$ . Also, the predicate **comp-coord** is an abbreviation for the fact that the translation of  $v$ 's coordinates to the computation plane are  $v'_x, v'_y$  and  $u_t$ . The exact formula expressing this can be found in the proof of Lemma 4.17, when the translation of relation predicates is explained.

- (iii) For the set  $Type_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R) = \{\alpha \in (\mathcal{A}_{st}, \mathcal{A}_t) \mid \alpha(R) = \mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(R)\}$ , we compute two separate relations storing the  $\mathcal{A}_t$ -type, respectively  $\mathcal{A}_{st}$ -type of the encoded relation. In the previous formula, it was checked whether there exists a transformation mapping all points in  $R$  to points in the formula coded by  $m$  (that is, in  $F$ ). Here, we compute that transformation:

$$\begin{aligned}
T_{\mathcal{A}_t} := & \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_\alpha, v_\beta) | (\mathbf{TCoSys}_{\mathcal{A}}(u_{t_O}, u_{t_E}) \wedge \\
& \mathbf{CoSys}_{\mathcal{A}}^n(u_0, u_1, u_2)) \rightarrow (\exists w)(N(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w) \wedge \\
& (\forall v_x)(\forall v_y)(\forall v_t)(F(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w, v_x, v_y, v_t) \leftrightarrow \\
& (\exists v)(\exists v'_x)(\exists v'_y)(\exists v'_t)(\exists v''_x)(\exists v''_y)(\exists v_{\alpha_{0,0}})(\exists v_{\alpha_{0,1}})(\exists v_{\alpha_{1,0}})(\exists v_{\alpha_{1,1}})(\exists v_{\beta_0})(\exists v_{\beta_1}) \\
& (R(v) \wedge \mathbf{comp-coord}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v, v'_x, v'_y, v'_t) \wedge \\
& \psi_{\mathcal{A}_{st}}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_{\alpha_{0,0}}, v_{\alpha_{0,1}}, v_{\alpha_{1,0}}, v_{\alpha_{1,1}}, v_{\beta_0}, v_{\beta_1}, \\
& v'_t, v'_x, v'_y, v'_t, v''_x, v''_y, v'_t) \wedge \\
& \psi_{\mathcal{A}_t}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_\alpha, v_\beta, v''_x, v''_y, v'_t, v_x, v_y, v_t))\}
\end{aligned}$$

and

$$\begin{aligned}
T_{\mathcal{A}_{st}} := & \{(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_{\alpha_{0,0}}, v_{\alpha_{0,1}}, v_{\alpha_{1,0}}, v_{\alpha_{1,1}}, v_{\beta_0}, v_{\beta_1}, u_t) | \\
& (\mathbf{TCoSys}_{\mathcal{A}}(u_{t_O}, u_{t_E}) \wedge \mathbf{CoSys}_{\mathcal{A}}^n(u_0, u_1, u_2)) \rightarrow \\
& (\exists w)(N(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w) \wedge (\forall v_x)(\forall v_y)(\forall v_t) \\
& (F(u_{t_O}, u_{t_E}, u_0, u_1, u_2, w, v_x, v_y, v_t) \leftrightarrow (\exists v)(\exists v'_x)(\exists v'_y)(\exists v'_t)(\exists v_\alpha)(\exists v_\beta) \\
& (R(v) \wedge \mathbf{comp-coord}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v, v'_x, v'_y, v'_t) \wedge \\
& T_{\mathcal{A}_t}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_\alpha, v_\beta) \wedge \\
& \psi_{\mathcal{A}_t}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_\alpha, v_\beta, v'_x, v'_y, v'_t, v'_x, v'_y, v_t) \wedge \\
& \psi_{\mathcal{A}_{st}}(u_{t_O}, u_{t_E}, u_0, u_1, u_2, v_{\alpha_{0,0}}, v_{\alpha_{0,1}}, v_{\alpha_{1,0}}, v_{\alpha_{1,1}}, v_{\beta_0}, v_{\beta_1}, \\
& v_t, v'_x, v'_y, v_t, v_x, v_y, v_t))\}.
\end{aligned}$$

The decoding algorithm can be expressed in  $\text{FO}(\Pi) + \text{while}$ . Input databases are encoded by natural numbers. A counter machine simulates the query on this natural number and returns a natural number that encodes the output. In the decoding algorithm, again all natural numbers are enumerated and the evaluation of the terms and formulas they encode are stored in relations called  $T$  and  $F$ . When the number that is the output of the counter machine is encountered, the relation  $F$  contains all points of the result, up to the transformation stored in  $\text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}$  (because the query is  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic). The result corresponds to the set  $Q(\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F}))$ . As  $Q$  is assumed to be a  $(\mathcal{A}_{st}, \mathcal{A}_t)$ -generic query, we have that for all  $f \in \text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})$

$$Q(\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})) = Q(f(\mathcal{F})) = f(Q(\mathcal{F})),$$

so  $Q(\mathcal{F})$  is computed as

$$\bigcup_{f \in \text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})} f^{-1}(Q(\mathbf{Canon}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F}))) = \bigcup_{f \in \text{Type}_{(\mathcal{A}_{st}, \mathcal{A}_t)}(\mathcal{F})} f^{-1}(f(Q(\mathcal{F}))).$$

For completeness, we give a program `Decode` that, when applied to the encoding  $N_\varphi$  of a formula  $\varphi$ , computes in a relation variable `Dec` the spatio-temporal relation defined by  $\varphi$ . Thereto it suffices to modify the encode program as shown in here. The input is a natural number encoding a relation.

```

m := 0;
T := ∅;
F := ∅;

```

```

found :=False;
while not found do
   $m := m + 1$ ;
  build relations  $T$  and  $F$ ;
  found:=  $m = N_\varphi$ ;
end while
Dec := all points which are the image under the transformation stored in
Type $_{(A_{st}, A_t)}$  of the points with coordinates
(represented as points on the line  $u_0u_1$ )  $p_x, p_y, p_t$ 
such that  $F(u_{t_0}, u_{t_E}, u_0, u_1, u_2, m, p_1, p_2, p_t)$ ;

```

The formula constructing the output, using the above, only differs slightly from the formulas we gave when explaining the encoding algorithm. In the encoding phase, it had to be checked, for some natural number  $m$ , whether there existed a transformation mapping all points of  $R$  to the points satisfying the formula encoded by  $m$ . Also, that transformation was computed. Here, we have the transformation stored in  $\text{Type}_{(A_{st}, A_t)}$ , and we know we have the right natural number  $m$ , so all points mapped by the transformation in  $\text{Type}_{(A_{st}, A_t)}$  to points satisfying the formula encoded by  $m$ , are returned.

To conclude we summarize the conversion procedure. Given a  $k$ -ary computable query  $Q$  over a schema  $\sigma = \{R_1, \dots, R_m\}$ , there exists a counter program  $M$  such that for each database  $\mathcal{F}$  over  $\sigma$ , if  $(n_{R_1}, \dots, n_{R_m})$  are the results of applying the program Encode to  $\mathcal{F}$  then  $M(n_{R_1}, \dots, n_{R_m})$  is the encoding of the quantifier-free formula defining  $Q(\mathcal{F})$ , using the variables  $x_1^1, \dots, x_1^{n+1}, \dots, x_K^1, \dots, x_K^{n+1}$ . If  $Q(\mathcal{F})$  is not defined, then  $M$  does not halt on this input. As already noted above, we can simulate  $M$  by a program  $P$  in  $\text{FO}(\Pi) + \text{while}$ . Hence, the query  $Q$  is expressed by the program

```

Encode;
P;
Decode;

```

This concludes the proof of Theorem 5.10.

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#### REFERENCES

- ABEL, D. AND OOI, B. C., Eds. 1993. *Advances in Spatial Databases—3rd Symposium (SSD'93)*. Lecture Notes in Computer Science, vol. 692. Springer-Verlag, New York.
- ALLEN, J. F. 1983. Maintaining knowledge about temporal intervals. *Commun. ACM* 26, 11, 832–843.
- ALLEN, J. F. AND FERGUSON, G. 1994. Actions and events in interval temporal logic. *J. Logic Comput.* 4, 5, 531–579.
- BOCHNAK, J., COSTE, M., AND ROY, M. 1987. *Géométrie Algébrique Réelle*. Springer-Verlag, Berlin, Germany.
- BÖHLEN, M. H., JENSEN, C. S., AND SCHOLL, M., Eds. 1999. *Proceedings of the International Workshop on Spatio Temporal Database Management (STDBM'99)*. Lecture Notes in Computer Science, vol. 1678. Springer-Verlag, New York.

- BUCHMANN, A., ED. 1989. *Design and Implementation of Large Spatial Databases—1st Symposium (SSD'89)*. Lecture Notes in Computer Science, vol. 409. Springer-Verlag, New York.
- CHANDRA, A. K. AND HAREL, D. 1980. Computable queries for relational data bases. *J. Comp. Syst. Sci.* 21, 2, 156–178.
- CHEN, C. X. AND ZANIOLO, C. 2000. SQLST: A spatio-temporal data model and query language. In *Conceptual Modeling, 19th International Conference on Conceptual Modeling (ER'00)*, A. H. F. Laender, S. W. Liddle, and V. C. Storey, Eds. Lecture Notes in Computer Science, vol. 1920. Springer-Verlag, New York, 96–111.
- CHOMICKI, J. AND REVESZ, P. 1997. Constraint-based interoperability of spatio-temporal databases. In *Proceedings of the 5th International Symposium on Advances in Spatial Databases*. Lecture Notes in Computer Science, vol. 1262. Springer-Verlag, New York, 142–161.
- CHOMICKI, J. AND REVESZ, P. 1999. A geometric framework for specifying spatiotemporal objects. In *Proceedings of the 6th International Workshop on Temporal Representation and Reasoning*. IEEE Computer Society, Press, Los Alamitos, CA, 41–46.
- DESLOGE, E. 1982. *Classical Mechanics*. Wiley, New York.
- EGENHOFER, M. J. AND HERRING, J. R., EDs. 1995. *Advances in Spatial Databases—4th Symposium (SSD'95)*. Lecture Notes in Computer Science, vol. 951. Springer-Verlag, New York.
- ERWIG, M., GÜTING, R. H., SCHNEIDER, M., AND VAZIRGIANNIS, M. 1999. Spatio-temporal data types: An approach to modeling and querying moving objects in databases. *GeoInformatica* 3, 3, 269–296.
- FRANK, A., GRUMBACH, S., GÜTING, R., JENSEN, C., KOUBARAKIS, M., LORENTZOS, N., MANOPOULOS, Y., NARDELLI, E., PERNICI, B., SCHEK, H.-J., SCHOLL, M., SELLIS, T., THEODOULIDIS, B., AND WIDMAYER, P. 1999. Chorochronos: A research network for spatiotemporal database systems. *SIGMOD Record* 28, 12–21.
- GRUMBACH, S., RIGAUX, P., AND SEGOUFIN, L. 1998. Spatio-temporal data handling with constraints. In *Proceedings of the 6th International Symposium on Advances in Geographic Information Systems (ACM-GIS'98)*, R. Laurini, K. Makki, and N. Pissinou, Eds. ACM, New York, 106–111.
- GRUMBACH, S. AND SU, J. 1997. Queries with arithmetical constraints. *Theor. Comput. Sci.* 173, 1, 151–181.
- GUNTHER, O. AND SCHEK, H.-J., EDs. 1991. *Advances in Spatial Databases—2nd Symposium SSD'91*. Lecture Notes in Computer Science, vol. 525. Springer-Verlag, New York.
- GÜTING, R., ED. 1999. *Advances in Spatial Databases—6th Symposium (SSD'99)*. Lecture Notes in Computer Science, vol. 1651. Springer-Verlag, New York.
- GÜTING, R. AND SCHNEIDER, M. 2005. *Moving Object Databases*. Morgan Kaufmann, San Francisco, CA.
- GÜTING, R. H., BOHLEN, M. H., ERWIG, M., JENSEN, C. S., LORENTZOS, N. A., SCHNEIDER, M., AND VAZIRGIANNIS, M. 2000. A foundation for representing and querying moving objects. *ACM Trans. Datab. Syst.* 25, 1–42.
- GYSSENS, M., DEN BUSSCHE, J. V., AND GUCHT, D. V. 1999. Complete geometric query languages. *J. Comput. Syst. Sci.* 58, 3, 483–511.
- KANELLAKIS, P. C., KUPER, G. M., AND REVESZ, P. 1990. Constraint query languages. In *Proceedings of the 9th ACM Symposium on Principles of Database Systems (PODS'90)*. ACM, New York, 299–313.
- KANELLAKIS, P. C., KUPER, G. M., AND REVESZ, P. 1995. Constraint query languages. *J. Comput. Syst. Sci.* 51, 26–52.
- KULJPERS, B., PAREDAENS, J., AND GUCHT, D. V. 2000. Towards a theory of movie database queries. In *Proceedings of the 7th International Workshop on Temporal Representation and Reasoning*. IEEE Computer Society, Press, Los Alamitos, CA, 95–102.
- KUPER, G. AND SCHOLL, M. 2000. Geographic information systems. In *Constraint Databases*, J. Paredaens, G. Kuper, and L. Libkin, Eds. Springer-Verlag, New York, Chapter 2, 175–198.
- LIBKIN, L. 2000. Variable independence, quantifier elimination, and constraint representation. In *Automata, Languages and Programming, 27th International Colloquium (ICALP 2000)*, U. Montanari, J. D. P. Rolim, and E. Welzl, Eds. Lecture Notes in Computer Science, vol. 1853. Springer-Verlag, New York, 260–271.
- MATIJASEVICH, Y. V. 1993. *Hilbert's Tenth Problem*. MIT Press, Cambridge, MA.



- PAREDAENS, J., KUPER, G., AND LIBKIN, L., Eds. 2000. *Constraint databases*. Springer-Verlag, New York.
- PAREDAENS, J., VAN DEN BUSSCHE, J., AND VAN GUCHT, D. 1994. Towards a theory of spatial database queries. In *Proceedings of the 13th ACM Symposium on Principles of Database Systems*. ACM, New York, 279–288.
- PFOSE, D. AND TRYFONA, N. 1998. Requirements, definitions and notations for spatiotemporal application environments. In *Proceedings of the 6th International Symposium on Advances in Geographic Information Systems (ACM-GIS'98)*, R. Laurini, K. Makki, and N. Pissinou, Eds. ACM, New York, 124–130.
- REVESZ, P. 2002. *Introduction to Constraint Databases*. Springer-Verlag, New York.
- RIGAUX, P., SCHOLL, M., AND VOISARD, A. 2000. *Introduction to Spatial Databases: Applications to GIS*. Morgan Kaufmann, Reading, MA.
- SCHOLL, M. AND VOISARD, A., Eds. 1997. *Advances in Spatial Databases—5th Symposium (SSD'97)*. Lecture Notes in Computer Science, vol. 1262. Springer-Verlag, New York.
- SCHWABHÄUSER, W., SZMIELEW, W., AND TARSKI, A. 1983. *Metamathematische Methoden in der Geometrie*. Springer-Verlag, New York.
- SISTLA, A. P., WOLFSON, O., CHAMBERLAIN, S., AND DAO, S. 1997. Modeling and querying moving objects. In *Proceedings of the 13th International Conference on Data Engineering*. IEEE Computer Society, Press, Los Alamitos, CA, 422–432.
- SU, J., XU, H., AND IBARRA, O. 2001. Moving objects: Logical relationships and queries. In *Advances in Spatial and Temporal Databases (SSTD'01)*. Lecture Notes in Computer Science, vol. 2121. Springer-Verlag, New York, 3–19.
- VAN DEN BUSSCHE, J. 2000. Constraint Databases, queries and query languages. In *Constraint Databases*, J. Paredaens, G. Kuper, and L. Libkin, Eds. Springer-Verlag, New York, Chapter 2, 21–54.
- WOLFSON, O. 2002. Moving objects information management: The database challenge. In *Proceedings of the 5th International Workshop (NGITS)*. Springer-Verlag, New York, 75–89.

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