Superposition of Markov sources and Long Range Dependence

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Abstract
This paper introduces a model to study the phenomenon of long range dependence. This model consists of an infinite superposition of independent Markovian ON/OFF–sources. A condition for assuring long range dependence is given and the Hurst parameter together with the correlation decay is derived for a specific example. We also give a physical interpretation of the existing long range dependence by means of the Ising model.

Keywords
ATM, Ising Model, Long Range Dependence, Markov Sources, Phase Transitions

1 INTRODUCTION

Recent measurements on Ethernet traffic (see e.g. [LTWW93a,b,94]) show that its profile exhibits Long Range Dependent (LRD) characteristics. Also for variable bit rate video traffic a similar behaviour has been observed (see [BSTW95]). LRD means that correlations extend to an infinite time scale and the correlation decay follows a power law. Traditional finite state Markovian traffic models, such as Markov Modulated Poission Processes (MMPP), Markovian Arrival Processes (MAP), etc..., have an exponential correlation decay and can therefore not adequately model this type of ATM data traffic. These observations have triggered new research activities on models which are able to capture LRD characteristics. Several approaches have been proposed in literature. The theories of Fractional Brownian Motion [Nor94], chaotic maps [Pru95] and regularly varying functions (see [Box96]) have have been successfuly applied to study LRD properties. An alternative approach consists of modeling LRD over a chosen time scale by using a Markovian approximation (see [AJN95]). Recently, several authors (see [DB97],[LTG95], etc...) use

the superposition of an infinite number of ON/OFF–sources to characterize LRD traffic.
This paper follows the last approach. We consider a class of processes consisting of the superposition of an infinite number of ON/OFF–sources. Through the characterisation of the sum of the covariances, it is possible to establish a simple explicit necessary and sufficient condition for the process to be LRD. This condition expresses an asymptotical eigenvalue degeneracy of the transition matrix. The simplicity of the model allows to derive explicit formulas for the powerlaw correlation decay and the Hurst parameter. It is also shown that under LRD conditions the mean queue length is infinite.

In physics, long range dependence occurs in turbulence, quantum field theory, $1/f$ noises, and critical phenomena. For instance, in the two–dimensional Ising model, a critical phenomenon appears. The Ising model is a model for ferromagnetism, and describes phase transitions. A phase transition is e.g. a transition from non aligned spins to aligned spins (magnetism). This transition from an unordered system to an ordered one, happens at the critical point (e.g. a certain temperature). At this point, the correlation decay between spin regions goes from an exponential decay to a powerlaw decay. The eigenvalue degeneracy of the transfer matrices (see later) is also here a necessary condition for the existence of long range dependence. It appears that our model can be incorporated in the Ising model.
More research is needed to study the possibility of applying known techniques from statistical physics to our model, and the question of a physical interpretation of the queueing process in this context remains open.

This paper is structured as follows. In the next section, the single ON/OFF–source is described and the eigenvalue structure of the transition matrix is related to the correlation structure. This is then generalized to a finite superposition of sources of this type. At the end of this section we introduce our model. In Section 3 we prove a necessary and sufficient condition for the long range dependence of this model, and two examples are given. As a direct consequence, we prove in Section 4 the infiniteness of the mean queue length, and discuss some problems concerning the queueing behaviour. In Section 5, the exponent of the power decay of the correlation is derived, and an explicit formula is given for the Hurst parameter. We introduce a physical counterpart, namely the Ising model, of our model in Section 6 and we observe the connection between a phase transition and long range dependence. Conclusions are drawn in Section 8.

2 CORRELATION AND EIGENVALUE STRUCTURE

An ON/OFF–source is defined to be a two–state discrete time Markov chain. In the OFF state the process generates 0 cells/slot, and in the ON state the process generates 1 cells/slot. The duration of the ON state is geometrically distributed with parameter β. Similarly the duration of the OFF state is geometrically distributed with parameter α. Let $X = \{X_i\}_{i\in\mathbb{N}}$ be the two–state discrete time Markov chain with irreducible and aperiodic transition matrix $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

The stationary probability vector is given by $\boldsymbol{\pi} = (\pi_0, \pi_1) = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta}\right)$, and the arrival rate of this process is $\lambda = \frac{1-\alpha}{2-\alpha-\beta}$. The covariance $\gamma(k) = E[X_i, X_{i+k}] - E[X_i]E[X_{i+k}]$ is given by

$$\begin{aligned} \gamma(k) &= \gamma^k \pi_0 \pi_1 \\ &= \gamma^k \frac{(1-\alpha)(1-\beta)}{(2-\alpha-\beta)^2}, \end{aligned}$$

where $\gamma = \alpha + \beta - 1$ is the second largest eigenvalue of $\mathbf{P}$. As $\mathbf{P}$ is stochastic and irreducible, it follows that $\gamma < 1$, and hence $\lim_{k\to\infty} \gamma(k) = 0$.
Let $X^{(\ell)}$ be N such Markovian ON/OFF–source with transition matrix $\mathbf{P}^{(\ell)}$ and arrival rate $\lambda^{(\ell)}$, $\ell = 1, 2, \ldots, N$. We consider the superposition of these N sources, $Y_i^{(N)} = \sum_{\ell=1}^{N} X_i^{(\ell)}$. Since each ON/OFF–source can be viewed as a D-MAP, and the superposition of a finite number of D-MAP's is a D-BMAP [Blon92, BG97], we conclude that the corresponding transition matrix $\mathbf{P}_N$, is given by the Kronecker product

$$\mathbf{P}_N = \bigotimes_{\ell=1}^{N} \mathbf{P}^{(\ell)}.$$

Because the ON/OFF–sources are independent, the covariance $\gamma_N(k)$ of the superposed sources equals

$$\gamma_N(k) = \sum_{\ell=1}^{N} \left(\gamma^{(\ell)}\right)^k \frac{(1-\alpha_\ell)(1-\beta_\ell)}{(2-\alpha_\ell-\beta_\ell)^2}.$$

The queueing model used in what follows is the D-BMAP/D/1–queue. The average number of cells arriving in the queue is given by

$$\lambda^{(N)} = \sum_{\ell=1}^{N} \pi_1^{(\ell)}.$$

We assume that $\lambda^{(N)} < 1$ to ensure the existence of a stochastic equilibrium for the queueing system.
Let us now consider an infinite superposition of ON/OFF–sources. Denote $Y_i^{(\infty)} = \sum_{\ell=1}^{\infty} X_i^{(\ell)}$, $\mathbf{P}_\infty = \lim_{N\to\infty} \mathbf{P}_N$, $\gamma_\infty(k) = \lim_{N\to\infty} \gamma_N(k)$, and the arrival rate $\lambda^{(\infty)} = \sum_{\ell=1}^{\infty} \lambda^{(\ell)}$. We shall derive some properties of $Y^{(\infty)} = \{Y_i^{(\infty)}\}_{i\in\mathbb{N}}$ in the next sections.

The sequence $Y_1^{(\infty)}, Y_2^{(\infty)}, \ldots$ of stationary random variables is called long range dependent if

$$\sum_{k=1}^{\infty} \mathrm{Cov}\left(Y_1^{(\infty)}, Y_k^{(\infty)}\right) = \infty$$

(See [Ber94, RMV96]). For our model we have to ensure that

$$\sum_{k=1}^{\infty} \gamma_\infty(k) = \infty.$$

Proposition 1 *A superposition of an infinite number of Markovian ON/OFF–sources $Y^{(\infty)}$ is long range dependent if and only if*

$$\sum_{\ell=1}^{\infty} \frac{1-\alpha_\ell}{(1-\beta_\ell)^2} = \infty, \tag{1}$$

where α_ℓ and β_ℓ are the elements of $\mathbf{P}^{(\ell)}$.

Proof. We must proof that the series

$$\sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} \left(\gamma^{(\ell)}\right)^k \frac{(1-\alpha_\ell)(1-\beta_\ell)}{(2-\alpha_\ell-\beta_\ell)^2} \tag{2}$$

diverges. We shall prove that the series of condition (1) has the same divergent behaviour as (2). For this we need two observations
Firstly the stability condition $\lambda^{(\infty)} < 1$ implies that $\lim_{\ell\to\infty} \frac{1-\alpha_\ell}{2-\alpha_\ell-\beta_\ell} = 0$. This means that there exist an M' such that for $\ell \gg M'$,

$$1-\epsilon < 1 - \frac{1-\alpha_\ell}{2-\alpha_\ell-\beta_\ell} < 1$$

for a fixed ϵ.
Secondly the inequality

$$\sum_{\ell=1}^{\infty} \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}} \frac{(1-\alpha_\ell)(1-\beta_\ell)}{(2-\alpha_\ell-\beta_\ell)^2} < \max_\ell \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}}$$

implies that long range dependency exists only if $\sup_\ell \gamma^{(\ell)} = 1$. If $\gamma^{(m)} = 1$ for a finite $m < \infty$, then matrix $\mathbf{P}^{(\infty)}$ is of the form

$$\bigotimes_{\ell=1}^{m-1} \mathbf{P}^{(\ell)} \otimes \begin{pmatrix} \bigotimes_{\ell=m+1}^{\infty} \mathbf{P}^{(\ell)} & \mathbf{O} \\ \mathbf{O} & \bigotimes_{\ell=m+1}^{\infty} \mathbf{P}^{(\ell)} \end{pmatrix},$$

and the model consist of two seperate and identical submodels. Hence, we assume that $\lim_{\ell\to\infty} \gamma^{(\ell)} = 1$. This implies that $\exists M''$ such that for $\ell \gg M''$

$$1-\epsilon < \gamma^{(\ell)} < 1$$

for the same ϵ. We let $M = \max\{M', M''\}$. For $\ell \gg M$, we have

$$(1-\epsilon)^2 \frac{(1-\alpha_\ell)}{(2-\alpha_\ell-\beta_\ell)(1-\gamma^{(\ell)})} < \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}} \frac{(1-\alpha_\ell)(1-\beta_\ell)}{(2-\alpha_\ell-\beta_\ell)^2} < \frac{(1-\alpha_\ell)}{(2-\alpha_\ell-\beta_\ell)(1-\gamma^{(\ell)})}.$$

In view of

$$\frac{1-\alpha_\ell}{2-\alpha_\ell-\beta_\ell} = \frac{\frac{1-\alpha_\ell}{1-\beta_\ell}}{\frac{1-\alpha_\ell}{1-\beta_\ell}+1},$$

and,

$$\begin{aligned} \frac{(1-\alpha_\ell)}{(2-\alpha_\ell-\beta_\ell)(1-\gamma^{(\ell)})} &= \frac{1-\alpha_\ell}{(2-\alpha_\ell-\beta_\ell)^2} \\ &= \frac{1-\alpha_\ell}{(1-\beta_\ell)^2(\frac{1-\alpha_\ell}{1-\beta_\ell}+1)^2}, \end{aligned}$$

and the fact that,

$$\lim_{\ell\to\infty} \frac{\frac{1-\alpha_\ell}{1-\beta_\ell}}{\frac{1-\alpha_\ell}{1-\beta_\ell}+1} = 0 \Leftrightarrow \lim_{\ell\to\infty} \frac{1-\alpha_\ell}{1-\beta_\ell} = 0,$$

the following bounds are obtained,

$$(1-\epsilon)^2 \frac{1}{2} \frac{1-\alpha_\ell}{(1-\beta_\ell)^2} \leq \frac{1-\alpha_\ell}{(1-\beta_\ell)^2(\frac{1-\alpha_\ell}{1-\beta_\ell}+1)^2} \leq \frac{1-\alpha_\ell}{(1-\beta_\ell)^2},$$

for $\ell \gg M$. This shows that the series (1) and (2) have the same divergent behaviour, and concludes the proof. ■

From this Proposition it follows that for appropriate choices of α_ℓ and β_ℓ, the resulting superposition of heterogeneous Markovian ON/OFF–sources is LRD.
We now give two examples

Example 1
Let (see [DB97])

$$\mathbf{P}_\ell = \begin{pmatrix} 1-(1/a)^\ell & (1/a)^\ell \\ (b/a)^\ell & 1-(b/a)^\ell \end{pmatrix}$$

with $1 < b < a$. It is clear that $\gamma^{(\ell)} = 1-(1/a)^\ell - (b/a)^\ell$ goes to 1 as ℓ tends to infinity. We see that the model $\mathbf{P}_\infty$ is LRD iff the series

$$\sum_{\ell=1}^{\infty} \left(\frac{a}{b^2}\right)^\ell$$

diverges, or iff $b^2 \leq a$.

Example 2
Let

$$\mathbf{P}_\ell = \begin{pmatrix} 1-1/\ell^p & 1/\ell^p \\ 1/\ell^q & 1-1/\ell^q \end{pmatrix},$$

for $\ell = 2, 3, \ldots$. To ensure $\rho < 1$ we need $p > q + 2$. It is clear that $\gamma^{(\ell)} = 1 - 1/\ell^p - 1/\ell^q$ goes to 1 as ℓ tends to infinity. We see that the model $\mathbf{P}_\infty$ is LRD iff the series

$$\sum_{\ell=2}^{\infty} \left(\frac{1}{\ell^{p-2q}} \right)$$

diverges, or iff $p \leq 2q + 1$.

4 QUEUEING BEHAVIOUR

The mean queue length $\bar{L}$ of an infinite superposition of Markovian ON/OFF–sources is given by (see Chapter 6 in [Neu89])

$$\bar{L} = \rho + \frac{1}{1-\rho} \sum_{\ell=1}^{\infty} \sum_{k>\ell} \pi_1^{(\ell)} \pi_1^{(k)} \left(1 + \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}} + \frac{\gamma^{(k)}}{1-\gamma^{(k)}} \right). \qquad (3)$$

The following Proposition is a direct consequence of Proposition 1.

Proposition 2 *The mean queue length $\bar{L}$ of an infinite superposition of Markovian ON/OFF–sources is infinite if and only if it is long range dependent.*

Proof. First assume that the arrival process $Y^{(\infty)}$ is LRD and consider the following term of (3),

$$\sum_{\ell=1}^{\infty} \sum_{k>\ell} \pi_1^{(\ell)} \pi_1^{(k)} \frac{\gamma^{(k)}}{1-\gamma^{(k)}}.$$

Interchanging the summation indices and using similar bounds as in Proposition 1, it is clear that

$$\sum_{\ell=2}^{\infty} \frac{1-\alpha_\ell}{(1-\beta_\ell)^2} \pi_1^{(1)} \leq \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell} \pi_1^{(k)} \pi_1^{(\ell)} \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}} \leq \rho \sum_{\ell=2}^{\infty} \frac{1-\alpha_\ell}{(1-\beta_\ell)^2},$$

and hence by Proposition 1, $\bar{L} = \infty$.
Now, let $\bar{L} = \infty$. Because the term in expression (3)

$$\sum_{\ell=1}^{\infty} \sum_{k>i} \pi_1^{(\ell)} \pi_1^{(k)} \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}} \leq \rho \sum_{\ell=1}^{\infty} \frac{1-\alpha_\ell}{(1-\beta_\ell)^2},$$

is bounded, it immediately follows that

$$\sum_{\ell=1}^{\infty} \sum_{k>i} \pi_1^{(\ell)} \pi_1^{(k)} \frac{\gamma^{(\ell)}}{1-\gamma^{(\ell)}} = \infty,$$

implies long range dependence of $Y^{(\infty)}$. This proves the Proposition. ■

More interesting properties of the queue distribution, like e.g. the tail of the queue length distribution, have not yet been derived for our model. The eigenvalue degeneracy of $\mathbf{P}_\infty$ induces severe difficulties. In the absence of this degeneracy, one can rely on the dominant pole approximation[LB97]. In this case there is an unique isolated dominant pole which governs the asymptotic behaviour of the queue [ACW94, Falk94, Miegh96]. The eigenvalue degeneracy transforms the isolated pole into a accumulation point in the complex plane. As a consequence there is not a single dominating pole, but an infinite number of poles which have to be taken into account.

Using large deviation techniques, Buffet and Duffield [BD92, DZ93, Duff92, Duff93] derived a bound for the loss probability. This method does not seem to be applicable to our model.

Boxma [Box96] has shown that in a special case,

$$\mathrm{P}(\tau_A = m) \sim m^{-\beta} \Rightarrow \Pr(U > m) \sim m^{-(\beta-2)}$$

holds, using the Fluid Flow approach. We currently belief that this is also true for our model, but have been unable to establish this result (see also [LB97]) for the superposition of Markovian ON/OFF–sources.

5 THE CORRELATION DECAY, HURST PARAMETER AND INDEX OF DISPERSION FOR COUNTS

In this section we give a more detailed study of example 2. The method follows a similar reasoning as for example 1 (see [DB97]).

5.1 Correlation Decay and Hurst parameter

Proposition 3 *The correlation decay of the arrival process $Y^{(\infty)}$ of example 2 is given by*

$$Cov(Y_1^{(\infty)}, Y_k^{(\infty)}) \sim k^{\frac{q-p+1}{q}}, \qquad (4)$$

for large k.

PROOF. We need to find the decay of the series

$$\sum_{i=2}^{\infty}(1 - \frac{1}{i^p} - \frac{1}{i^q})^k \frac{i^{(q+p)}}{(i^p + i^q)^2}.$$

Observe that the second factor can be bounded by

$$\frac{1}{3}\frac{i^{q+p}}{i^{2p}} \leq \frac{i^{q+p}}{(i^p + i^q)^2} \leq \frac{i^{q+p}}{i^{2p}}.$$

A second simplification of (4) is is done by replacing $(1-\frac{1}{i^p}-\frac{1}{i^q})^k$ by $(1-\frac{1}{i^q})^k$,

$$\sum_{i=2}^{\infty}(1-\frac{1}{i^p}-\frac{1}{i^q})^k i^{(q-p)} \leq \sum_{i=2}^{\infty}(1-\frac{1}{i^q})^k i^{(q-p)}.$$

Because $(1-\frac{1}{i^q})^k i^{(q-p)}$ is a nonnegative descending function, we use a contineous variable x instead of i and we apply Cauchy's integral test,

$$\int_2^{\infty}(1-\frac{1}{x^q})^k x^{(q-p)}dx \leq \sum_{i=2}^{\infty}(1-\frac{1}{i^q})^k i^{(q-p)} \leq (1-\frac{1}{2^q})^k 2^{(q-p)}+\int_2^{\infty}(1-\frac{1}{x^q})^k x^{(q-p)}dx.$$

To evaluate this integral, we use the inequalities

$$(1-\frac{1}{x^q})^k \leq e^{-\frac{k}{x^q}} = \frac{1}{e^{\frac{k}{x^q}}} \leq \frac{x^q}{k}.$$

Furthermore, it is clear that

$$1-k\frac{1}{x^q} \leq (1-\frac{1}{x^q})^k \leq 1-k\frac{1}{x^q}+\frac{k^2}{2}\left(\frac{1}{x^q}\right)^2.$$

We want that $1-k\frac{1}{x^q}>0$ so it is sufficient that $x>k^{\frac{1}{q}}$. We can now give an upper bound,

$$\int_2^{\infty}(1-\frac{1}{x^q})^k x^{(q-p)}dx \leq \int_2^{k^{1/q}}\frac{x^{2q-p}}{k}dx+\int_{k^{1/q}}^{\infty}\left(1-k\frac{1}{x^q}+\frac{k^2}{2}\left(\frac{1}{x^q}\right)^2\right)x^{q-p}dx.$$

The last part equals

$$\frac{1}{2q-p+1}\left(k^{\frac{q-p+1}{q}}-\frac{2^{2q-p+1}}{k}\right)-\left(\frac{1}{q-p+1}+\frac{1}{1-p}+\frac{1}{2(1-q-p)}\right)k^{\frac{q-p+1}{q}}.$$

Using $1-k\frac{1}{x^q}$ as under bound and after some similar calculations we find the bounds

$$C_1 k^{\frac{q-p+1}{q}} \leq \sum_{i=2}^{\infty}(1-\frac{1}{i^q})^k i^{(q-p)} \leq C_2 k^{\frac{q-p+1}{q}},$$

where C_1 and C_2 are some constants. We now have to assure that this bounds also the original sum. For this it is neccesary to bound

$$\sum_{i=2}^{\infty}\left((1-\frac{1}{i^q})^k-(1-\frac{1}{i^p}-\frac{1}{i^q})^k\right)i^{(q-p)}. \tag{5}$$

We have that

$$(1-\frac{1}{i^q})^k-(1-\frac{1}{i^p}-\frac{1}{i^q})^k \leq k\left(1-\frac{1}{i^q}\right)^{k-1}\frac{1}{i^p}.$$

Using similar techniques as above, it is possible to show that the difference (5) is bounded by $Ck^{\frac{q-p+1}{q}-\delta}$, with C a constant and $\delta>0$. The resulting powerlaw decay is $k^{\frac{q-p+1}{q}}$. This proves the Proposition. ■

The degree of long range dependence is often expressed by means of the Hurst parameter.

Proposition 4 *The Hurst parameter for the discrete time arrival process $Y^{(\infty)}$ of example 2 is given by*

$$H = \frac{3q - p + 1}{q} \tag{6}$$

PROOF. The Hurst parameter can be derived from the power decay of the covariance [RMV96]. If the power of the covariance decay is $k^{-\beta}$, then the Hurst parameter is given by $H = \frac{2-\beta}{2}$. It directly follows from Proposition 4 that $\beta = -\frac{q-p+1}{q}$, hence the Hurst parameter for example 2 is

$$H = \frac{3q - p + 1}{2q}.$$

From the conditions $p > q + 1$ and $p < 2q + 1$, it follows that $H \in (\frac{1}{2}, 1]$. ■
For completeness, we mention that for example 1, the powerlaw decay is given by $k^{-\frac{\log b}{\log b - \log a}}$, and hence $H = \frac{\left(2 - \frac{\log b}{\log b - \log a}\right)}{2}$.

5.2 The Index of Dispersion for Counts

In this section we derive an expression for the limit of the Index of Dispersion for Counts (IDC) of the process $Y^{(\infty)}$.
Denote N_k the number of arrivals in an interval of length k. The *Index of Dispersion for Counts* (IDC) at time k is defined to be the variance of the number of arrivals in an interval of length k divided by the the mean number of arrivals in this interval, i.e.

$$I(k) = \frac{\mathrm{Var}(N_k)}{\mathrm{E}(N_k)}.$$

Denote $I^{(\ell)}(k)$ the IDC of the process $X^{(\ell)}$ with $\lim_{k \to \infty} I^{(\ell)}(k) = J^{(\ell)}$ and $I^{(\infty)}(k)$ the IDC of the process $Y^{(\infty)}$, with $\lim_{k \to \infty} I^{(\infty)}(k) = J^{(\infty)}$.
From [BG97], we know that

$$J^{(\ell)} = \frac{\boldsymbol{\pi}^{(\ell)} \mathbf{P}_1^{(\ell)} \mathbf{e} - 3[\boldsymbol{\pi}^{(\ell)} \mathbf{P}_1^{(\ell)} \mathbf{e}]^2 + 2\boldsymbol{\pi}^{(\ell)} \mathbf{P}_1^{(\ell)} \mathbf{Z}^{(\ell)} \mathbf{P}_1^{(\ell)} \mathbf{e}}{\boldsymbol{\pi}^{(\ell)} \mathbf{P}_1^{(\ell)} \mathbf{e}}, \tag{7}$$

with $\mathbf{Z}^{(\ell)}$ the fundamental matrix of the Markov chain $\mathbf{P}^{(\ell)}$, given by

$$\mathbf{Z}^{(\ell)} = [\mathbf{I} - (\mathbf{P}^{(\ell)} - \mathbf{e}\boldsymbol{\pi}^{(\ell)})]^{-1},$$

and $\mathbf{P}_1^{(\ell)}$ given by

$$\mathbf{P}_1^{(\ell)} = \begin{pmatrix} 0 & 0 \\ 1/\ell^q & 1-1/\ell^q \end{pmatrix}.$$

Furthermore,

$$\lim_{k\to\infty} I^{(\infty)}(k) = \frac{\sum_{\ell=1}^{\infty}[\lambda^{(\ell)} - 3(\lambda^{(\ell)})^2 + 2\boldsymbol{\pi}^{(\ell)}\mathbf{P}_1^{(\ell)}\mathbf{Z}^{(\ell)}\mathbf{P}_1^{(\ell)}\mathbf{e}]}{\sum_{\ell=1}^{\infty}\lambda^{(\ell)}}. \tag{8}$$

It is easy to show that

$$\mathbf{Z}^{(\ell)} = \frac{1}{(\ell^p+\ell^q)^2}\begin{pmatrix} \ell^p(\ell^p+\ell^q+\ell^{2q}) & \ell^q(\ell^p+\ell^q-\ell^{p+q}) \\ \ell^p(\ell^p+\ell^q-\ell^{p+q}) & \ell^q(\ell^p+\ell^q+\ell^{2p}) \end{pmatrix}.$$

Hence,

$$\boldsymbol{\pi}^{(\ell)}\mathbf{P}_1^{(\ell)}\mathbf{Z}^{(\ell)}\mathbf{P}_1^{(\ell)}\mathbf{e} = \frac{\ell^q}{(\ell^p+\ell^q)^3}\,[\ell^{2q} - \ell^{2p} + \ell^{2p+q}].$$

Using this expression in (8), we obtain that

$$J^{(\infty)} = \frac{\lambda^{(\infty)} - 3\sum_{\ell=1}^{\infty}(\lambda^{(\ell)})^2 + 2\sum_{\ell=1}^{\infty}\frac{\ell^q[\ell^{2q}-\ell^{2p}+\ell^{2p+q}]}{(\ell^p+\ell^q)^3}}{\lambda^{(\infty)}}. \tag{9}$$

From equation (9) it follows that the limit of the IDC of the process $Y^{(\infty)}$ is infinite if $p \leqslant 2q+1$, which is exactly the condition under which the process has the long range dependence property. This is in agreement with the criterion that a process is long range dependent if its IDC is diverging.

6 CORRESPONDENCES BETWEEN LRD IN TELECOMMUNICATION AND PHASE TRANSITIONS IN STATISTICAL PHYSICS

There is an important similarity between our model and a model of phase transitions in statistical physics, namely the Ising model. This model was introduced in 1925 by Ising [Isin25] as a model for ferromagnetism, and is solved analytically by Onsager in 1944 [Ons44, Kau49, KO49, SML64].

We consider electrons, located on a rectangular lattice, who can have two different spins, spin up or spin down. with each microscopic configuration $\mathcal{O} = \{\omega(i,j) = \text{up/down} \mid 1 \le i \le n, 1 \le j \le m\}$, one associates a probability [Geo88, KSK76, PF91]

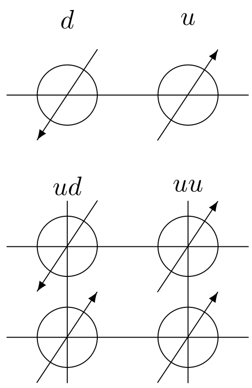
$$P(\mathcal{O}) = \frac{1}{Z}e^{E(\mathcal{O})},$$

where $E(\mathcal{O})$ is a function, called the interaction energy, and where $Z = \sum_{\mathcal{O}} \exp E(\mathcal{O})$ is called the partition function. One whishes to deduce macroscopic statistical properties, by taking the thermodynamic limit, i.e. expanding the lattice to the whole plane.

Focussing on the classical Ising model, the interaction energy is of the form

$$E(\mathcal{O}) = -j \sum_{i=1}^{m-1} \sum_{j} \omega(i,j)\omega(i+1,j) - j \sum_{i=1}^{m-1} \sum_{j} \omega(i,j)\omega(i,j+1)$$

which is clearly nearest–neighbour, i.e. the summation is over nearest–neighbour points on the lattice.
The calculation of the macroscopic properties can be done in an elegant way using transfer matrices. The principle is to put the values of the interaction energy in a matrix (see Figure 1).



$$\begin{pmatrix} E(uu) & \mathbf{E(ud)} \\ E(du) & E(dd) \end{pmatrix}$$

$$\begin{pmatrix} E(uuuu) & \mathbf{E(uuud)} & E(uudu) & E(uudd) \\ E(uduu) & E(udud) & E(uddu) & E(uddd) \\ E(duuu) & E(duud) & E(dudu) & E(dudd) \\ E(dduu) & E(ddud) & E(dddu) & E(dddd) \end{pmatrix}$$

Figure 1 Transfer matrices for the one– and two–dimensional Ising model. The bold entries correspond to the shown spin configuration

One also assumes periodic horizontal boundary conditions. This means that for a row of size n, the $n+1$th spin equals the first spin. In this way the partition function is nothing but the trace of these matrices. Remark that for a $1 \times m$ lattice , the size of the corresponding transfer matrix is $2^m \times 2^m$. The transfer matrix of three electrons in a single row is given by the product of the single 2×2 transfer matrix. It can now be proven that, when the number of rows increases to infinity, the correlation function of two spins in different column, decays to zero according to a power law, if the transfer matrix is asymptotically degenerate, i.e. the second greatest eigenvalue equals the greatest eigenvalue!
To conclude this section, we simplify the Ising model by assuming no vertical interactions. The corresponding transfer matrix is then the Kronecker product of the 2×2 transfer matrix, corresponding to a single row. It is now clear that Figure 2 establishes the link with our model.

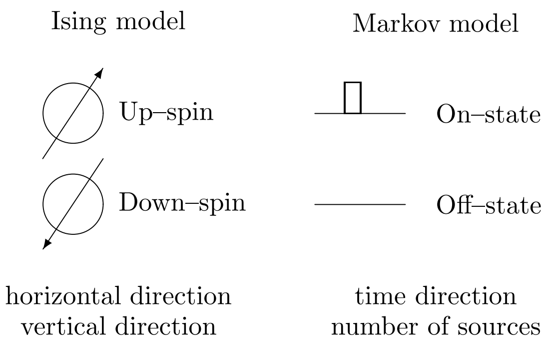


Figure 2 Correspondences between the Ising Model and the Arrival Process $Y^{(\infty)}$

Of course the Ising model is far more complex, admitting vertical interactions (dependent sources). In the thermodynamic limit the infinite transfer matrix of the Ising model is asymptotic degenerate below a certain value of j. Moreover j is dependent on the temperature T, so beneath the critical temperature T_c, the system has long range order. For our model, we constructed the transfer (transition) matrices of the rows (sources) in such a way that that the resulting infinite transfer matrix is always asymptotic degenerate (see Proposition 1). Nevertheless, we can view condition (1) as a way of determining an abstract critical point. If we take e.g. example 2, we can fix q and take p as 'temperature'. The critical value is then $p_c = 2q + 1$ (see Figure 3).

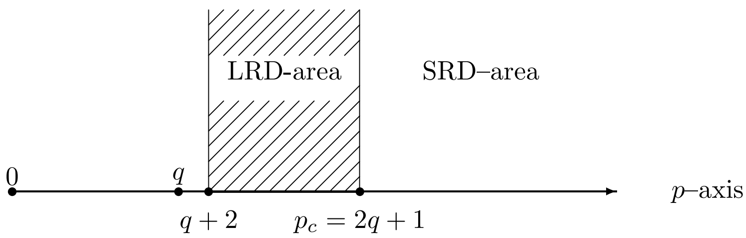


Figure 3 The phase diagram of example 2.

For a more accurate description of the Ising model see [DG72, Thom72]

7 CONCLUSIONS AND FUTURE WORK

In this paper we proved a necessary and sufficient condition for long range dependence of an infinite superposition of heterogeneous Markovian ON/OFF–sources. Two simple examples are given and the corresponding Hurst parameters are derived. We are currently investigating the existence of a general method for calculating the Hurst parameter of our model. It is also shown that for our model, long range dependence directly implies infinite mean queue length. The characterization of the queue length distribution remains open. We give a physical interpretation of our model by means of the Ising model. The existence of long range dependence is a consequence of the asymptotical degeneracy of the transition matrix. Further research is needed to study the possibility of applying known techniques of statistical physics to the context of ATM modelling. The admittance of dependent sources (vertical interactions) seems a first step. Also an interpretation of the queueing process in terms of the Ising model, could be of great help in deriving the tail of the queue length distribution.

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